

FINAL JEE-MAIN EXAMINATION – MARCH, 2021

(Held On Thursday 18th March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS

SECTION-A

1. Let $y = y(x)$ be the solution of the differential

$$\text{equation } \frac{dy}{dx} = (y+1)\left((y+1)e^{x^{2/2}} - x\right), 0 < x < 2.1,$$

with $y(2) = 0$. Then the value of $\frac{dy}{dx}$ at $x = 1$ is equal to :

- (1) $\frac{-e^{3/2}}{(e^2 + 1)^2}$ (2) $-\frac{2e^2}{(1 + e^2)^2}$
 (3) $\frac{e^{5/2}}{(1 + e^2)^2}$ (4) $\frac{5e^{1/2}}{(e^2 + 1)^2}$

Official Ans. by NTA (1)

Sol. Let $y + 1 = Y$

$$\therefore \frac{dY}{dx} = Y^2 e^{\frac{x^2}{2}} - xY$$

$$\text{Put } -\frac{1}{Y} = k$$

$$\Rightarrow \frac{dk}{dx} + k(-x) = e^{\frac{x^2}{2}}$$

$$\text{I.F.} = e^{-\frac{x^2}{2}}$$

$$\therefore k = (x + c)e^{x^{2/2}}$$

$$\text{Put } k = -\frac{1}{y+1}$$

$$\therefore y + 1 = -\frac{1}{(x + c)e^{x^{2/2}}} \dots(i)$$

when $x = 2, y = 0$, then $c = -2 - \frac{1}{e^2}$

differentiate equation (i) & put $x = 1$

$$\text{we get } \left(\frac{dy}{dx}\right)_{x=1} = -\frac{e^{3/2}}{(1 + e^2)^2}$$

TEST PAPER WITH SOLUTION

2. In a triangle ABC, if $|\overline{BC}| = 8, |\overline{CA}| = 7,$

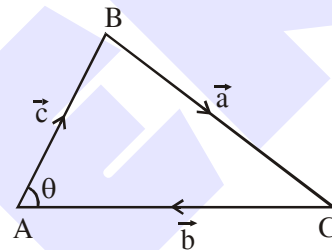
$$|\overline{AB}| = 10, \text{ then the projection of the vector } \overline{AB}$$

on \overline{AC} is equal to :

- (1) $\frac{25}{4}$ (2) $\frac{85}{14}$ (3) $\frac{127}{20}$ (4) $\frac{115}{16}$

Official Ans. by NTA (2)

Sol.



$$|\vec{a}| = 8, |\vec{b}| = 7, |\vec{c}| = 10$$

$$\cos \theta = \frac{|\vec{b}|^2 + |\vec{c}|^2 - |\vec{a}|^2}{2|\vec{b}||\vec{c}|} = \frac{17}{28}$$

Projection of \vec{c} on \vec{b}

$$= |\vec{c}| \cos \theta$$

$$= 10 \times \frac{17}{28}$$

$$= \frac{85}{14}$$

3. Let the system of linear equations

$$4x + \lambda y + 2z = 0$$

$$2x - y + z = 0$$

$$\mu x + 2y + 3z = 0, \lambda, \mu \in \mathbb{R}.$$

has a non-trivial solution. Then which of the following is true ?

- (1) $\mu = 6, \lambda \in \mathbb{R}$ (2) $\lambda = 2, \mu \in \mathbb{R}$
 (3) $\lambda = 3, \mu \in \mathbb{R}$ (4) $\mu = -6, \lambda \in \mathbb{R}$

Official Ans. by NTA (1)

Sol. For non-trivial solution

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 2\mu - 6\lambda + \lambda\mu = 12$$

when $\mu = 6$, $12 - 6\lambda + 6\lambda = 12$
which is satisfied by all λ

4. Let $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ be defined by

$$f(x) = \frac{x-2}{x-3}$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given as

$$g(x) = 2x - 3$$

Then, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$ is equal to

- (1) 7 (2) 2 (3) 5 (4) 3

Official Ans. by NTA (3)

Sol. $f(x) = y = \frac{x-2}{x-3}$

$$\therefore x = \frac{3y-2}{y-1}$$

$$\therefore f^{-1}(x) = \frac{3x-2}{x-1}$$

& $g(x) = y = 2x - 3$

$$\therefore x = \frac{y+3}{2}$$

$$\therefore g^{-1}(x) = \frac{x+3}{2}$$

$$\therefore f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$$

$$\therefore x^2 - 5x + 6 = 0 \begin{cases} x_1 \\ x_2 \end{cases}$$

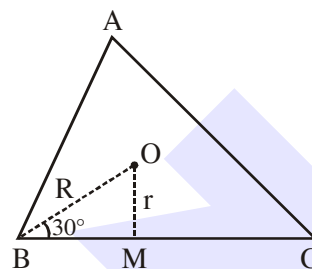
\therefore sum of roots
 $x_1 + x_2 = 5$

5. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x + y = 3$. If R and r be the radius of circumcircle and incircle respectively of ΔABC , then $(R + r)$ is equal to :

- (1) $\frac{9}{\sqrt{2}}$ (2) $7\sqrt{2}$ (3) $2\sqrt{2}$ (4) $3\sqrt{2}$

Official Ans. by NTA (1)

Sol.



$$r = OM = \frac{3}{\sqrt{2}}$$

$$\& \sin 30^\circ = \frac{1}{2} = \frac{r}{R} \Rightarrow R = \frac{6}{\sqrt{2}}$$

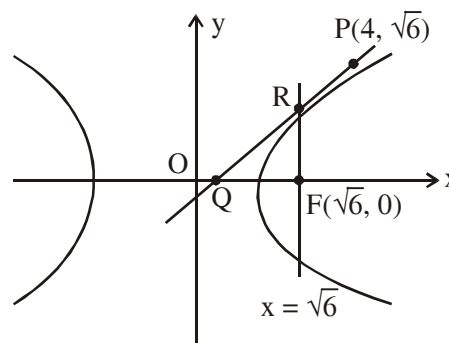
$$\therefore r + R = \frac{9}{\sqrt{2}}$$

6. Consider a hyperbola $H : x^2 - 2y^2 = 4$. Let the tangent at a point $P(4, \sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1, y_1)$, $x_1 > 0$. If F is a focus of H which is nearer to the point P , then the area of ΔQFR is equal to

- (1) $4\sqrt{6}$ (2) $\sqrt{6} - 1$
(3) $\frac{7}{\sqrt{6}} - 2$ (4) $4\sqrt{6} - 1$

Official Ans. by NTA (3)

Sol.



$$\frac{x^2}{4} - \frac{y^2}{2} = 1$$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$$

$$\therefore \text{Focus } F(ae, 0) \Rightarrow F(\sqrt{6}, 0)$$

equation of tangent at P to the hyperbola is

$$2x - y\sqrt{6} = 2$$

tangent meet x-axis at Q(1, 0)

$$\& \text{ latus rectum } x = \sqrt{6} \text{ at } R\left(\sqrt{6}, \frac{2}{\sqrt{6}}(\sqrt{6}-1)\right)$$

$$\therefore \text{Area of } \Delta_{QFR} = \frac{1}{2}(\sqrt{6}-1) \cdot \frac{2}{\sqrt{6}}(\sqrt{6}-1)$$

$$= \frac{7}{\sqrt{6}} - 2$$

7. If P and Q are two statements, then which of the following compound statement is a tautology ?

- (1) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$
 (2) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$
 (3) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$
 (4) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

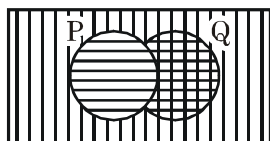
Official Ans. by NTA (2)

Sol. LHS of all the options are same i.e.

$$\begin{aligned} (P \rightarrow Q) \wedge \sim Q &\equiv (\sim P \vee Q) \wedge \sim Q \\ &\equiv (\sim P \wedge \sim Q) \vee (Q \wedge \sim Q) \\ &\equiv \sim P \wedge \sim Q \end{aligned}$$

(A) $(\sim P \wedge \sim Q) \rightarrow Q$
 $\equiv \sim(\sim P \wedge \sim Q) \vee Q$
 $\equiv (P \vee Q) \vee Q \neq \text{tautology}$

(B) $(\sim P \wedge \sim Q) \rightarrow \sim P$
 $\equiv \sim(\sim P \wedge \sim Q) \vee \sim P$
 $\equiv (P \vee Q) \vee \sim P$



\Rightarrow Tautology

(C) $(\sim P \wedge \sim Q) \rightarrow P$

$$\begin{aligned} &\equiv (P \vee Q) \vee P \neq \text{Tautology} \\ \text{(D)} \quad &(\sim P \wedge \sim Q) \rightarrow (P \wedge Q) \\ &\equiv (P \vee Q) \vee (P \wedge Q) \neq \text{Tautology} \end{aligned}$$

Aliter :

P	Q	$P \vee Q$	$P \wedge Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	T	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	T	T

8. Let $g(x) = \int_0^x f(t) dt$, where f is continuous

function in $[0, 3]$ such that $\frac{1}{3} \leq f(t) \leq 1$ for all

$t \in [0, 1]$ and $0 \leq f(t) \leq \frac{1}{2}$ for all $t \in (1, 3]$.

The largest possible interval in which $g(3)$ lies is :

- (1) $\left[-1, -\frac{1}{2}\right]$ (2) $\left[-\frac{3}{2}, -1\right]$
 (3) $\left[\frac{1}{3}, 2\right]$ (4) $[1, 3]$

Official Ans. by NTA (3)

Sol. $\frac{1}{3} \leq f(t) \leq 1 \forall t \in [0, 1]$

$$0 \leq f(t) \leq \frac{1}{2} \forall t \in (1, 3]$$

Now, $g(3) = \int_0^3 f(t) dt = \int_0^1 f(t) dt + \int_1^3 f(t) dt$

$$\therefore \int_0^1 \frac{1}{3} dt \leq \int_0^1 f(t) dt \leq \int_0^1 1 dt \quad \dots(1)$$

$$\text{and } \int_1^3 0 dt \leq \int_1^3 f(t) dt \leq \int_1^3 \frac{1}{2} dt \quad \dots(2)$$

Adding, we get

$$\frac{1}{3} + 0 \leq g(3) \leq 1 + \frac{1}{2}(3-1)$$

$$\frac{1}{3} \leq g(3) \leq 2$$

9. Let S_1 be the sum of first $2n$ terms of an arithmetic progression. Let S_2 be the sum of first $4n$ terms of the same arithmetic progression. If $(S_2 - S_1)$ is 1000, then the sum of the first $6n$ terms of the arithmetic progression is equal to:
 (1) 1000 (2) 7000 (3) 5000 (4) 3000
Official Ans. by NTA (4)

Sol. $S_{2n} = \frac{2n}{2} [2a + (2n - 1)d]$, $S_{4n} = \frac{4n}{2} [2a + (4n - 1)d]$

$$\Rightarrow S_2 - S_1 = \frac{4n}{2} [2a + (4n - 1)d] - \frac{2n}{2} [2a + (2n - 1)d]$$

$$= 4an + (4n - 1)2nd - 2na - (2n - 1)dn$$

$$= 2na + nd[8n - 2 - 2n + 1]$$

$$\Rightarrow 2na + nd[6n - 1] = 1000$$

$$2a + (6n - 1)d = \frac{1000}{n}$$

Now, $S_{6n} = \frac{6n}{2} [2a + (6n - 1)d]$

$$= 3n \cdot \frac{1000}{n} = 3000$$

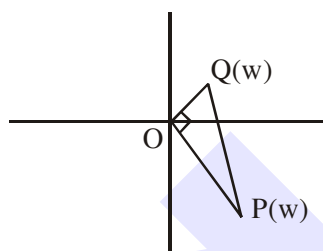
10. Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to :
 (1) 4 (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) 2

Official Ans. by NTA (2)

Sol. $w = 1 - \sqrt{3}i \Rightarrow |w| = 2$

Now, $|z| = \frac{1}{|w|} \Rightarrow |z| = \frac{1}{2}$

and $\arg(z) = \frac{\pi}{2} + \arg(w)$



$$\Rightarrow \text{Area of triangle} = \frac{1}{2} \cdot OP \cdot OQ$$

$$= \frac{1}{2} \cdot 2 \cdot \frac{1}{2} = \frac{1}{2}$$

11. Let in a series of $2n$ observations, half of them are equal to a and remaining half are equal to $-a$. Also by adding a constant b in each of these observations, the mean and standard deviation of new set become 5 and 20, respectively. Then the value of $a^2 + b^2$ is equal to :
 (1) 425 (2) 650 (3) 250 (4) 925
Official Ans. by NTA (1)

Sol. Let observations are denoted by x_i for $1 \leq i < 2n$

$$\bar{x} = \frac{\sum x_i}{2n} = \frac{(a + a + \dots + a) - (a + a + \dots + a)}{2n}$$

$$\Rightarrow \bar{x} = 0$$

$$\text{and } \sigma_x^2 = \frac{\sum x_i^2}{2n} - (\bar{x})^2 = \frac{a^2 + a^2 + \dots + a^2}{2n} - 0 = a^2$$

$$\Rightarrow \sigma_x = a$$

Now, adding a constant b then $\bar{y} = \bar{x} + b = 5$

$$\Rightarrow b = 5$$

and $\sigma_y = \sigma_x$ (No change in S.D.) $\Rightarrow a = 20$

$$\Rightarrow a^2 + b^2 = 425$$

12. Let $S_1 : x^2 + y^2 = 9$ and $S_2 : (x - 2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points :

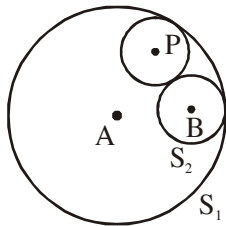
- (1) $(0, \pm\sqrt{3})$ (2) $\left(\frac{1}{2}, \pm\frac{\sqrt{5}}{2}\right)$
 (3) $\left(2, \pm\frac{3}{2}\right)$ (4) $(1, \pm 2)$

Official Ans. by NTA (3)

Sol. $S_1 : x^2 + y^2 = 9 \begin{cases} r_1 = 3 \\ A(0, 0) \end{cases}$

$S_2 : (x - 2)^2 + y^2 = 1 \begin{cases} r_2 = 1 \\ B(2, 0) \end{cases}$

$\therefore c_1 c_2 = r_1 - r_2$



\therefore given circle are touching internally

Let a variable circle with centre P and radius r

$\Rightarrow PA = r_1 - r$ and $PB = r_2 + r$

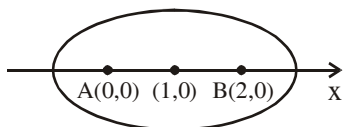
$\Rightarrow PA + PB = r_1 + r_2$

$\Rightarrow PA + PB = 4$ ($> AB$)

\Rightarrow Locus of P is an ellipse with foci at $A(0, 0)$ and $B(2, 0)$ and length of major axis is $2a = 4$,

$e = \frac{1}{2}$

\Rightarrow centre is at $(1, 0)$ and $b^2 = a^2(1 - e^2) = 3$ if x-ellipse



$\Rightarrow E: \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$

which is satisfied by $\left(2, \pm\frac{3}{2}\right)$

13. Let \vec{a} and \vec{b} be two non-zero vectors perpendicular to each other and $|\vec{a}| = |\vec{b}|$. If

$|\vec{a} \times \vec{b}| = |\vec{a}|$, then the angle between the vectors

$(\vec{a} + \vec{b} + (\vec{a} \times \vec{b}))$ and \vec{a} is equal to :

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(3) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (4) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$

Official Ans. by NTA (2)

Sol. $|\vec{a}| = |\vec{b}|$, $|\vec{a} \times \vec{b}| = |\vec{a}|$, $\vec{a} \perp \vec{b}$

$|\vec{a} \times \vec{b}| = |\vec{a}| \Rightarrow |\vec{a}||\vec{b}|\sin 90^\circ = |\vec{a}| \Rightarrow |\vec{b}| = 1 = |\vec{a}|$

\vec{a} and \vec{b} are mutually perpendicular unit vectors.

Let $\vec{a} = \hat{i}$, $\vec{b} = \hat{j} \Rightarrow \vec{a} \times \vec{b} = \hat{k}$

$\cos \theta = \frac{(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{i}}{\sqrt{3} \sqrt{1}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

14. Let in a Binomial distribution, consisting of 5 independent trials, probabilities of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to :

(1) $\frac{32}{625}$ (2) $\frac{80}{243}$ (3) $\frac{40}{243}$ (4) $\frac{128}{625}$

Official Ans. by NTA (1)

Sol. $P(X = 1) = {}^5C_1 \cdot p \cdot q^4 = 0.4096$

$P(X = 2) = {}^5C_2 \cdot p^2 \cdot q^3 = 0.2048$

$\Rightarrow \frac{q}{2p} = 2$

$\Rightarrow q = 4p$ and $p + q = 1$

$\Rightarrow p = \frac{1}{5}$ and $q = \frac{4}{5}$

Now

$P(X = 3) = {}^5C_3 \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^2 = \frac{10 \times 16}{125 \times 25} = \frac{32}{625}$

15. Let a tangent be drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$

at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $\theta \in (0, \frac{\pi}{2})$. Then the

value of θ such that the sum of intercepts on axes made by this tangent is minimum is equal to :

- (1) $\frac{\pi}{8}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

Official Ans. by NTA (3)

Sol. Equation of tangent be

$$\frac{x \cos \theta}{3\sqrt{3}} + \frac{y \sin \theta}{1} = 1, \quad \theta \in \left(0, \frac{\pi}{2}\right)$$

intercept on x-axis

$$OA = 3\sqrt{3} \sec \theta$$

intercept on y-axis

$$OB = \operatorname{cosec} \theta$$

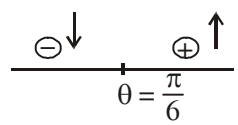
Now, sum of intercept

$$= 3\sqrt{3} \sec \theta + \operatorname{cosec} \theta = f(\theta) \text{ let}$$

$$f'(\theta) = 3\sqrt{3} \sec \theta \tan \theta - \operatorname{cosec} \theta \cot \theta$$

$$= 3\sqrt{3} \frac{\sin \theta}{\cos^2 \theta} - \frac{\cos \theta}{\sin^2 \theta}$$

$$= \frac{\cos \theta}{\sin^2 \theta} \cdot 3\sqrt{3} \left[\tan^3 \theta - \frac{1}{3\sqrt{3}} \right] = 0 \Rightarrow \theta = \frac{\pi}{6}$$



\Rightarrow at $\theta = \frac{\pi}{6}$, $f(\theta)$ is minimum

16. Define a relation R over a class of $n \times n$ real matrices A and B as "ARB iff there exists a non-singular matrix P such that $PAP^{-1} = B$ ".

Then which of the following is true ?

- (1) R is symmetric, transitive but not reflexive,
 (2) R is reflexive, symmetric but not transitive
 (3) R is an equivalence relation
 (4) R is reflexive, transitive but not symmetric

Official Ans. by NTA (3)

Sol. A and B are matrices of $n \times n$ order & ARB iff there exists a non singular matrix P ($\det(P) \neq 0$) such that $PAP^{-1} = B$

For reflexive

$$ARA \Rightarrow PAP^{-1} = A \quad \dots(1) \text{ must be true}$$

for $P = I$, Eq.(1) is true so 'R' is reflexive

For symmetric

$$ARB \Leftrightarrow PAP^{-1} = B \quad \dots(1) \text{ is true}$$

for BRA iff $PBP^{-1} = A \quad \dots(2) \text{ must be true}$

$$\therefore PAP^{-1} = B$$

$$P^{-1}PAP^{-1} = P^{-1}B$$

$$IAP^{-1}P = P^{-1}BP$$

$$A = P^{-1}BP \quad \dots(3)$$

from (2) & (3) $PBP^{-1} = P^{-1}BP$

can be true some $P = P^{-1} \Rightarrow P^2 = I$ ($\det(P) \neq 0$)

So 'R' is symmetric

For transitive

$$ARB \Leftrightarrow PAP^{-1} = B \dots \text{ is true}$$

$$BRC \Leftrightarrow PBP^{-1} = C \dots \text{ is true}$$

$$\text{now } PPAP^{-1}P^{-1} = C$$

$$P^2A(P^2)^{-1} = C \Rightarrow ARC$$

So 'R' is transitive relation

\Rightarrow Hence R is equivalence

17. A pole stands vertically inside a triangular park ABC. Let the angle of elevation of the top of

the pole from each corner of the park be $\frac{\pi}{3}$.

If the radius of the circumcircle of ΔABC is 2, then the height of the pole is equal to :

- (1) $\frac{2\sqrt{3}}{3}$ (2) $2\sqrt{3}$ (3) $\sqrt{3}$ (4) $\frac{1}{\sqrt{3}}$

Official Ans. by NTA (2)

Sol. Let $PD = h$, $R = 2$

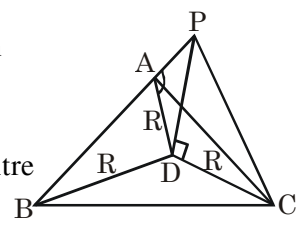
As angle of elevation

of top of pole from

A, B, C are equal So

D must be circumcentre

of ΔABC



$$\tan\left(\frac{\pi}{3}\right) = \frac{PD}{R} = \frac{h}{R}$$

$$h = R \tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

18. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha \in \mathbb{R}$, then the value of $27\sec^6\alpha + 8\operatorname{cosec}^6\alpha$ is equal to :
 (1) 350 (2) 500 (3) 400 (4) 250

Official Ans. by NTA (4)

Sol. $15\sin^4\alpha + 10\cos^4\alpha = 6$
 $15\sin^4\alpha + 10\cos^4\alpha = 6(\sin^2\alpha + \cos^2\alpha)^2$
 $(3\sin^2\alpha - 2\cos^2\alpha)^2 = 0$

$$\tan^2\alpha = \frac{2}{3}, \cot^2\alpha = \frac{3}{2}$$

$$\Rightarrow 27\sec^6\alpha + 8\operatorname{cosec}^6\alpha$$

$$= 27(\sec^6\alpha)^3 + 8(\operatorname{cosec}^6\alpha)^3$$

$$= 27(1 + \tan^2\alpha)^3 + 8(1 + \cot^2\alpha)^3$$

$$= 250$$

19. The area bounded by the curve $4y^2 = x^2(4-x)(x-2)$ is equal to :

(1) $\frac{\pi}{8}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{2}$ (4) $\frac{\pi}{16}$

Official Ans. by NTA (3)

Sol. $4y^2 = x^2(4-x)(x-2)$

$$|y| = \frac{|x|}{2} \sqrt{(4-x)(x-2)}$$

$$\Rightarrow y_1 = \frac{x}{2} \sqrt{(4-x)(x-2)}$$

and $y_2 = -\frac{x}{2} \sqrt{(4-x)(x-2)}$

D : $x \in [2, 4]$

Required Area

$$= \int_2^4 (y_1 - y_2) dx = \int_2^4 x \sqrt{(4-x)(x-2)} dx \quad \dots(1)$$

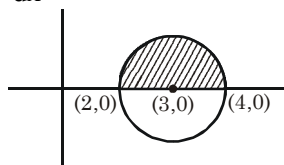
Applying $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\text{Area} = \int_2^4 (6-x) \sqrt{(4-x)(x-2)} dx \quad \dots(2)$$

(1) + (2)

$$2A = 6 \int_2^4 \sqrt{(4-x)(x-2)} dx$$

$$A = 3 \int_2^4 \sqrt{1-(x-3)^2} dx$$



$$A = 3 \cdot \frac{\pi}{2} \cdot 1^2 = \frac{3\pi}{2}$$

20. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x}, & \text{if } x < 0 \\ b, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}, & \text{if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then the value of $a + b$ is equal to :

(1) $-\frac{5}{2}$ (2) -2 (3) -3 (4) $-\frac{3}{2}$

Official Ans. by NTA (4)

- Sol.** $f(x)$ is continuous at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = f(0) = \lim_{x \rightarrow 0^-} f(x) \quad \dots(1)$$

$$f(0) = b \quad \dots(2)$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{\sin(a+1)x}{2x} + \frac{\sin 2x}{2x} \right)$$

$$= \frac{a+1}{2} + 1 \quad \dots(3)$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}$$

$$= \lim_{x \rightarrow 0^+} \frac{(x+bx^3-x)}{bx^{5/2}(\sqrt{x+bx^3} + \sqrt{x})}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x}(\sqrt{1+bx^2} + 1)} = \frac{1}{2} \quad \dots(4)$$

Use (2), (3) & (4) in (1)

$$\frac{1}{2} = b = \frac{a+1}{2} + 1$$

$$\Rightarrow b = \frac{1}{2}, a = -2$$

$$a + b = \frac{-3}{2}$$

SECTION-B

1. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to _____.

Official Ans. by NTA (0)

Sol. $P(x) = f(x^3) + xg(x^3)$
 $P(1) = f(1) + g(1) \dots(1)$
 Now $P(x)$ is divisible by $x^2 + x + 1$
 $\Rightarrow P(x) = Q(x)(x^2 + x + 1)$
 $P(w) = 0 = P(w^2)$ where w, w^2 are non-real cube roots of units
 $P(x) = f(x^3) + xg(x^3)$
 $P(w) = f(w^3) + wg(w^3) = 0$
 $f(1) + wg(1) = 2 \dots(2)$
 $P(w^2) = f(w^6) + w^2g(w^6) = 0$
 $f(1) + w^2g(1) = 0 \dots(3)$
 $(2) + (3)$
 $\Rightarrow 2f(1) + (w + w^2)g(1) = 0$
 $2f(1) = g(1) \dots(4)$
 $(2) - (3)$
 $\Rightarrow (w - w^2)g(1) = 0$
 $g(1) = 0 = f(1)$ from (4)
 from (1) $P(1) = f(1) + g(1) = 0$

2. Let I be an identity matrix of order 2×2 and

$P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$. Then the value of $n \in \mathbb{N}$ for

which $P^n = 5I - 8P$ is equal to _____.

Official Ans. by NTA (6)

Sol. $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$
 $5I - 8P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 16 & -8 \\ 40 & -24 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$
 $P^2 = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$
 $P^3 = \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix} \Rightarrow P^6 = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix} = P^n$
 $\Rightarrow n = 6$

3. If $\sum_{r=1}^{10} r!(r^3 + 6r^2 + 2r + 5) = \alpha(11!)$, then the value of α is equal to _____.

Official Ans. by NTA (160)

Sol. $\sum_{r=1}^{10} r! \{ (r+1)(r+2)(r+3) - 9(r+1) + 8 \}$
 $= \sum_{r=1}^{10} [\{ (r+3)! - (r+1)! \} - 8 \{ (r+1)! - r! \}]$
 $= (13! + 12! - 2! - 3!) - 8(11! - 1)$
 $= (12 \cdot 13 + 12 - 8) \cdot 11! - 8 + 8$
 $= (160)(11!)$
 Hence $\alpha = 160$

4. The term independent of x in the expansion of

$\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right]^{10}$, $x \neq 1$, is equal to _____.

Official Ans. by NTA (210)

Sol. $\left((x^{1/3} + 1) - \left(\frac{\sqrt{x} + 1}{\sqrt{x}} \right) \right)^{10}$
 $(x^{1/3} - x^{-1/2})^{10}$
 $T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$
 $\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0$
 $\Rightarrow r = 4$
 $T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$

5. Let $P(x)$ be a real polynomial of degree 3 which vanishes at $x = -3$. Let $P(x)$ have local minima at $x = 1$, local maxima at $x = -1$ and $\int_{-1}^1 P(x) dx = 18$, then the sum of all the coefficients of the polynomial $P(x)$ is equal to _____.

Official Ans. by NTA (8)

Sol. Let $p'(x) = a(x-1)(x+1) = a(x^2-1)$

$$p(x) = a \int (x^2-1) dx + c$$

$$= a \left(\frac{x^3}{3} - x \right) + c$$

Now $p(-3) = 0$

$$\Rightarrow a \left(-\frac{27}{3} + 3 \right) + c = 0$$

$$\Rightarrow -6a + c = 0 \quad \dots(1)$$

$$\text{Now } \int_{-1}^1 \left(a \left(\frac{x^3}{3} - x \right) + c \right) dx = 18$$

$$= 2c = 18 \Rightarrow c = 9 \quad \dots(2)$$

$$\Rightarrow \text{from (1) \& (2)} \Rightarrow -6a + 9 = 0 \Rightarrow a = \frac{3}{2}$$

$$\Rightarrow p(x) = \frac{3}{2} \left(\frac{x^3}{3} - x \right) + 9$$

sum of coefficient

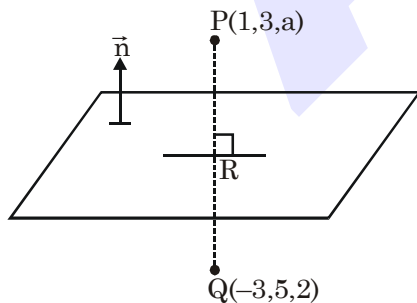
$$= \frac{1}{2} - \frac{3}{2} + 9$$

$$= 8$$

6. Let the mirror image of the point $(1, 3, a)$ with respect to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) - b = 0$ be $(-3, 5, 2)$. Then the value of $|a + b|$ is equal to _____.

Official Ans. by NTA (1)

Sol.



$$\text{plane} = 2x - y + z = b$$

$$R \equiv \left(-1, 4, \frac{a+2}{2} \right) \rightarrow \text{on plane}$$

$$\therefore -2 - 4 + \frac{a+2}{2} = b$$

$$\Rightarrow a + 2 = 2b + 12 \Rightarrow a = 2b + 10 \quad \dots(i)$$

$$\langle PQ \rangle = \langle 4, -2, a-2 \rangle$$

$$\therefore \frac{2}{4} = \frac{-1}{-2} = \frac{1}{a-2}$$

$$\Rightarrow a - 2 = 2 \Rightarrow a = 4, b = -3$$

$$\therefore |a + b| = 1$$

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) \neq 0$ for any $x \in \mathbb{R}$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(h) - 1) \text{ is equal to } \underline{\hspace{2cm}}.$$

Official Ans. by NTA (3)

Sol. If $f(x+y) = f(x) \cdot f(y)$ & $f'(0) = 3$ then

$$f(x) = a^x \Rightarrow f'(x) = a^x \cdot \ln a$$

$$\Rightarrow f'(0) = \ln a = 3 \Rightarrow a = e^3$$

$$\Rightarrow f(x) = (e^3)^x = e^{3x}$$

$$\lim_{x \rightarrow 0} \frac{f(x) - 1}{x} = \lim_{x \rightarrow 0} \left(\frac{e^{3x} - 1}{3x} \times 3 \right) = 1 \times 3 = 3$$

8. Let ${}^n C_r$ denote the binomial coefficient of x^r in the expansion of $(1+x)^n$.

$$\text{If } \sum_{k=0}^{10} (2^2 + 3k) {}^n C_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}, \alpha, \beta \in \mathbb{R},$$

then $\alpha + \beta$ is equal to _____.

Official Ans. by NTA (19)

Allen Answer (Bonus)

Sol. BONUS

Instead of ${}^n C_k$ it must be ${}^{10} C_k$ i.e.

$$\sum_{k=0}^{10} (2^2 + 3k) {}^{10} C_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$$

$$\text{LHS} = 4 \sum_{k=0}^{10} {}^{10} C_k + 3 \sum_{k=0}^{10} k \cdot \frac{10}{k} \cdot {}^9 C_{k-1}$$

$$= 4 \cdot 2^{10} + 3 \cdot 10 \cdot 2^9$$

$$= 19 \cdot 2^{10} = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$$

$$\Rightarrow \alpha = 0, \beta = 19 \Rightarrow \alpha + \beta = 19$$

9. Let P be a plane containing the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+5}{2}$ and parallel to the line $\frac{x-3}{4} = \frac{y-2}{-3} = \frac{z+5}{7}$. If the point $(1, -1, \alpha)$ lies on the plane P, then the value of $|5\alpha|$ is equal to _____.

Official Ans. by NTA (38)

Sol. Equation of plane is $\begin{vmatrix} x-1 & y+6 & z+5 \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix} = 0$

Now $(1, -1, \alpha)$ lies on it so

$$\begin{vmatrix} 0 & 5 & \alpha+5 \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix} = 0 \Rightarrow 5\alpha + 38 = 0 \Rightarrow |5\alpha| = 38$$

10. Let $y = y(x)$ be the solution of the differential equation $xdy - ydx = \sqrt{x^2 - y^2} dx$, $x \geq 1$, with $y(1) = 0$. If the area bounded by the line $x = 1$, $x = e^\pi$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to _____.

Official Ans. by NTA (4)

Sol. $xdy - ydx = \sqrt{x^2 - y^2} dx$
 $\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x} \sqrt{1 - \frac{y^2}{x^2}} dx$
 $\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

at $x = 1, y = 0 \Rightarrow c = 0$

$$y = x \sin(\ln x)$$

$$A = \int_1^{e^\pi} x \sin(\ln x) dx$$

$$x = e^t, dx = e^t dt \Rightarrow \int_0^\pi e^{2t} \sin(t) dt = A$$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5} (2 \sin t - \cos t)\right)_0^\pi = \frac{1 + e^{2\pi}}{5}$$

$$\alpha = \frac{1}{5}, \beta = \frac{1}{5} \text{ so } 10(\alpha + \beta) = 4$$