

**FINAL JEE-MAIN EXAMINATION – MARCH, 2021**

(Held On Wednesday 17<sup>th</sup> March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

**MATHEMATICS**

**TEST PAPER WITH SOLUTION**

**SECTION-A**

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = e^{-x} \sin x$ . If  $F : [0, 1] \rightarrow \mathbb{R}$  is a differentiable function

such that  $F(x) = \int_0^x f(t) dt$ , then the value of

$\int_0^1 (F'(x) + f(x)) e^x dx$  lies in the interval

(1)  $\left[\frac{327}{360}, \frac{329}{360}\right]$                       (2)  $\left[\frac{330}{360}, \frac{331}{360}\right]$

(3)  $\left[\frac{331}{360}, \frac{334}{360}\right]$                       (4)  $\left[\frac{335}{360}, \frac{336}{360}\right]$

**Official Ans. by NTA (2)**

**Sol.**  $f(x) = e^{-x} \sin x$

Now,  $F(x) = \int_0^x f(t) dt \Rightarrow F'(x) = f(x)$

$I = \int_0^1 (F'(x) + f(x)) e^x dx = \int_0^1 (f(x) + f(x)) \cdot e^x dx$

$= 2 \int_0^1 f(x) \cdot e^x dx = 2 \int_0^1 e^{-x} \sin x \cdot e^x dx$

$= 2 \int_0^1 \sin x dx$

$= 2(1 - \cos 1)$

$I = 2 \left\{ 1 - \left( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots \right) \right\}$

$I = 1 - \frac{2}{4} + \frac{2}{6} - \frac{2}{8} + \dots$

$1 - \frac{2}{4} < I < 1 - \frac{2}{4} + \frac{2}{6}$

$\frac{11}{12} < I < \frac{331}{360}$

$\Rightarrow I \in \left[ \frac{11}{12}, \frac{331}{360} \right]$

$\Rightarrow I \in \left[ \frac{330}{360}, \frac{331}{360} \right]$

**Ans. (2)**

2. If the integral  $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$ ,

where  $\alpha, \beta, \gamma$  are integers and  $[x]$  denotes the greatest integer less than or equal to  $x$ , then the value of  $\alpha + \beta + \gamma$  is equal to :

- (1) 0                      (2) 20                      (3) 25                      (4) 10

**Official Ans. by NTA (1)**

**Sol.** Let  $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

Function  $f(x) = \frac{[\sin 2\pi x]}{e^{\{x\}}}$  is periodic with period '1'  
Therefore

$I = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

$= 10 \int_0^1 \frac{[\sin 2\pi x]}{e^x} dx$

$= 10 \left( \int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x} dx \right)$

$= 10 \left( 0 + \int_{1/2}^1 \frac{(-1)}{e^x} dx \right)$

$= -10 \int_{1/2}^1 e^{-x} dx$

$= 10(e^{-1} - e^{-1/2})$

Now,

$10 \cdot e^{-1} - 10 \cdot e^{-1/2} = \alpha e^{-1} + \beta e^{-1/2} + \gamma$  (given)

$\Rightarrow \alpha = 10, \beta = -10, \gamma = 0$

$\Rightarrow \alpha + \beta + \gamma = 0$

**Ans. (1)**

3. Let  $y = y(x)$  be the solution of the differential equation  $\cos x (3\sin x + \cos x + 3)dy = (1 + y \sin x (3\sin x + \cos x + 3))dx$ ,  $0 \leq x \leq \frac{\pi}{2}$ ,  $y(0) = 0$ . Then,  $y\left(\frac{\pi}{3}\right)$  is equal to:

- (1)  $2 \log_e \left( \frac{2\sqrt{3} + 9}{6} \right)$       (2)  $2 \log_e \left( \frac{2\sqrt{3} + 10}{11} \right)$   
 (3)  $2 \log_e \left( \frac{\sqrt{3} + 7}{2} \right)$       (4)  $2 \log_e \left( \frac{3\sqrt{3} - 8}{4} \right)$

**Official Ans. by NTA (2)**

**Sol.**  $\cos x (3\sin x + \cos x + 3)dy = (1 + y \sin x (3\sin x + \cos x + 3))dx$

$$\frac{dy}{dx} - (\tan x)y = \frac{1}{(3\sin x + \cos x + 3)\cos x}$$

$$\text{I.F.} = e^{\int -\tan x dx} = e^{\ln|\cos x|} = |\cos x|$$

$$= \cos x \quad \forall x \in \left[0, \frac{\pi}{2}\right]$$

Solution of D.E.

$$y(\cos x) = \int (\cos x) \cdot \frac{1}{\cos x (3\sin x + \cos x + 3)} dx + C$$

$$y(\cos x) = \int \frac{dx}{3\sin x + \cos x + 3} dx + C$$

$$y(\cos x) = \int \frac{\left(\sec^2 \frac{x}{2}\right)}{2 \tan^2 \frac{x}{2} + 6 \tan \frac{x}{2} + 4} dx + C$$

Now

$$\text{Let } I_1 = \int \frac{\left(\sec^2 \frac{x}{2}\right)}{2 \left(\tan^2 \frac{x}{2} + 3 \tan \frac{x}{2} + 2\right)} dx + C$$

$$\text{Put } \tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt$$

$$I_1 = \int \frac{dt}{t^3 + 3t + 2} = \int \frac{dt}{(t+2)(t+1)}$$

$$= \int \left( \frac{1}{t+1} - \frac{1}{t+2} \right) dt$$

$$= \ln \left| \frac{t+1}{t+2} \right| = \ln \left| \frac{\tan \frac{x}{2} + 1}{\tan \frac{x}{2} + 2} \right|$$

So solution of D.E.

$$y(\cos x) = \ln \left| \frac{1 + \tan \frac{x}{2}}{2 + \tan \frac{x}{2}} \right| + C$$

$$\Rightarrow y(\cos x) = \ln \left( \frac{1 + \tan \frac{x}{2}}{2 + \tan \frac{x}{2}} \right) + C \quad \text{for } 0 \leq x < \frac{\pi}{2}$$

Now, it is given  $y(0) = 0$

$$\Rightarrow 0 = \ln \left( \frac{1}{2} \right) + C \Rightarrow \boxed{C = \ln 2}$$

$$\Rightarrow y(\cos x) = \ln \left( \frac{1 + \tan \frac{x}{2}}{2 + \tan \frac{x}{2}} \right) + \ln 2$$

$$\text{For } x = \frac{\pi}{3}$$

$$y\left(\frac{1}{2}\right) = \ln \left( \frac{1 + \frac{1}{\sqrt{3}}}{2 + \frac{1}{\sqrt{3}}} \right) + \ln 2$$

$$y = 2 \ln \left( \frac{2\sqrt{3} + 10}{11} \right)$$

**Ans.(2)**

4. The value of  $\sum_{r=0}^6 ({}^6C_r \cdot {}^6C_{6-r})$  is equal to :

- (1) 1124    (2) 1324    (3) 1024    (4) 924

**Official Ans. by NTA (4)**

**Sol.**  $\sum_{r=0}^6 {}^6C_r \cdot {}^6C_{6-r}$

$$= {}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + \dots + {}^6C_6 \cdot {}^6C_0$$

Now,

$$(1+x)^6 (1+x)^6$$

$$= ({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \dots + {}^6C_6x^6)$$

$$({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \dots + {}^6C_6x^6)$$

Comparing coefficient of  $x^6$  both sides

$${}^6C_0 \cdot {}^6C_6 + {}^6C_1 + {}^6C_5 + \dots + {}^6C_6 \cdot {}^6C_0 = {}^{12}C_6 = 924$$

**Ans.(4)**

5. The value of  $\lim_{n \rightarrow \infty} \frac{[r] + [2r] + \dots + [nr]}{n^2}$ , where  $r$  is non-zero real number and  $[r]$  denotes the greatest integer less than or equal to  $r$ , is equal to :

- (1)  $\frac{r}{2}$       (2)  $r$       (3)  $2r$       (4)  $0$

**Official Ans. by NTA (1)**

**Sol.** We know that

$$\begin{aligned} r &\leq [r] < r + 1 \\ \text{and } 2r &\leq [2r] < 2r + 1 \\ 3r &\leq [3r] < 3r + 1 \\ &\vdots \quad \vdots \quad \vdots \\ nr &\leq [nr] < nr + 1 \end{aligned}$$

$$\begin{aligned} r + 2r + \dots + nr \\ \leq [r] + [2r] + \dots + [nr] < (r + 2r + \dots + nr) + n \end{aligned}$$

$$\frac{n(n+1)}{2} \cdot r \leq \frac{[r] + [2r] + \dots + [nr]}{n^2} < \frac{n(n+1)}{2} r + n$$

Now,

$$\lim_{n \rightarrow \infty} \frac{n(n+1) \cdot r}{2 \cdot n^2} = \frac{r}{2}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)r}{2} + n}{n^2} = \frac{r}{2}$$

So, by Sandwich Theorem, we can conclude that

$$\lim_{n \rightarrow \infty} \frac{[r] + [2r] + \dots + [nr]}{n^2} = \frac{r}{2}$$

**Ans. (1)**

6. The number of solutions of the equation  $\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$ , for  $x \in [-1, 1]$ , and  $[x]$  denotes the greatest integer less than or equal to  $x$ , is :

- (1) 2                                      (2) 0  
(3) 4                                      (4) Infinite

**Official Ans. by NTA (2)**

**Sol.** Given equation

$$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$$

Now,  $\sin^{-1}\left[x^2 + \frac{1}{3}\right]$  is defined if

$$-1 \leq x^2 + \frac{1}{3} < 2 \Rightarrow \frac{-4}{3} \leq x^2 < \frac{5}{3}$$

$$\Rightarrow 0 \leq x^2 < \frac{5}{3} \quad \dots(1)$$

and  $\cos^{-1}\left[x^2 - \frac{2}{3}\right]$  is defined if

$$-1 \leq x^2 - \frac{2}{3} < 2 \Rightarrow \frac{-1}{3} \leq x^2 < \frac{8}{3}$$

$$\Rightarrow 0 \leq x^2 < \frac{8}{3} \quad \dots(2)$$

So, from (1) and (2) we can conclude

$$0 \leq x^2 < \frac{5}{3}$$

**Case - I** if  $0 \leq x^2 < \frac{2}{3}$

$$\sin^{-1}(0) + \cos^{-1}(-1) = x^2$$

$$\Rightarrow x + \pi = x^2$$

$$\Rightarrow x^2 = \pi$$

but  $\pi \notin \left[0, \frac{2}{3}\right)$

$\Rightarrow$  No value of 'x'

**Case - II** if  $\frac{2}{3} \leq x^2 < \frac{5}{3}$

$$\sin^{-1}(1) + \cos^{-1}(0) = x^2$$

$$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} = x^2$$

$$\Rightarrow x^2 = \pi$$

but  $\pi \notin \left[\frac{2}{3}, \frac{5}{3}\right)$

$\Rightarrow$  No value of 'x'

So, number of solutions of the equation is zero.

**Ans.(2)**

7. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at

even places be  $\frac{1}{2}$  and probability of

occurrence of 0 at the odd place be  $\frac{1}{3}$ . Then the probability that '10' is followed by '01' is equal to :

- (1)  $\frac{1}{18}$       (2)  $\frac{1}{3}$       (3)  $\frac{1}{6}$       (4)  $\frac{1}{9}$

**Official Ans. by NTA (4)**

**Sol.**  $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{2}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}$

or  $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{2}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}$

$\Rightarrow \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3}\right) + \left(\frac{2}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}\right)$

$\Rightarrow \frac{1}{9}$

8. The number of solutions of the equation

$x + 2 \tan x = \frac{\pi}{2}$  in the interval  $[0, 2\pi]$  is :

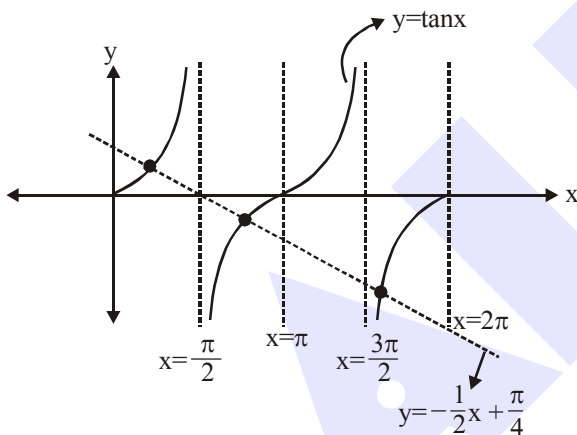
- (1) 3 (2) 4 (3) 2 (4) 5

**Official Ans. by NTA (1)**

**Sol.**  $x + 2 \tan x = \frac{\pi}{2}$

$\Rightarrow 2 \tan x = \frac{\pi}{2} - x$

$\Rightarrow \tan x = -\frac{1}{2}x + \frac{\pi}{4}$



Number of solutions of the given equation is '3'.

**Ans. (1)**

9. Let  $S_1, S_2$  and  $S_3$  be three sets defined as

$S_1 = \{z \in \mathbb{C} : |z - 1| \leq \sqrt{2}\}$

$S_2 = \{z \in \mathbb{C} : \operatorname{Re}((1 - i)z) \geq 1\}$

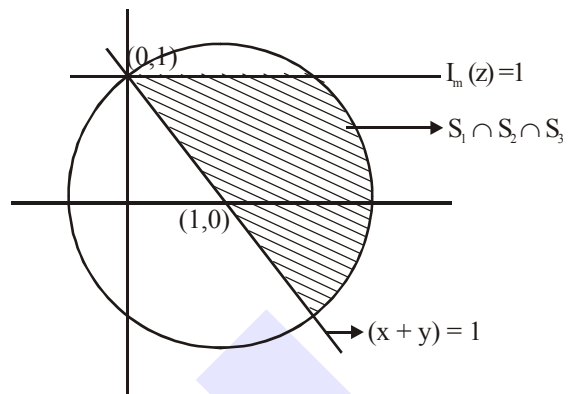
$S_3 = \{z \in \mathbb{C} : \operatorname{Im}(z) \leq 1\}$

Then the set  $S_1 \cap S_2 \cap S_3$

- (1) is a singleton  
 (2) has exactly two elements  
 (3) has infinitely many elements  
 (4) has exactly three elements

**Official Ans. by NTA (3)**

**Sol.** For  $|z - 1| \leq \sqrt{2}$ ,  $z$  lies on and inside the circle of radius  $\sqrt{2}$  units and centre  $(1, 0)$ .



**For  $S_2$**

Let  $z = x + iy$

Now,  $(1 - i)(z) = (1 - i)(x + iy)$

$\operatorname{Re}((1 - i)z) = x + y$

$\Rightarrow x + y \geq 1$

$\Rightarrow S_1 \cap S_2 \cap S_3$  has infinity many elements

**Ans. (3)**

10. If the curve  $y = y(x)$  is the solution of the differential equation

$2(x^2 + x^{5/4})dy - y(x + x^{1/4})dx = 2x^{9/4} dx$ ,  $x > 0$  which passes through the point

$\left(1, 1 - \frac{4}{3} \log_e 2\right)$ , then the value of  $y(16)$  is equal

to :

(1)  $4\left(\frac{31}{3} + \frac{8}{3} \log_e 3\right)$  (2)  $\left(\frac{31}{3} + \frac{8}{3} \log_e 3\right)$

(3)  $4\left(\frac{31}{3} - \frac{8}{3} \log_e 3\right)$  (4)  $\left(\frac{31}{3} - \frac{8}{3} \log_e 3\right)$

**Official Ans. by NTA (3)**

**Sol.**  $\frac{dy}{dx} - \frac{y}{2x} = \frac{x^{9/4}}{x^{5/4}(x^{3/4} + 1)}$

IF =  $e^{-\int \frac{dx}{2x}} = e^{-\frac{1}{2} \ln x} = \frac{1}{x^{1/2}}$

$y \cdot x^{-1/2} = \int \frac{x^{9/4} \cdot x^{-1/2}}{x^{5/4}(x^{3/4} + 1)} dx$

$\int \frac{x^{1/2}}{(x^{3/4} + 1)} dx$

$$x = t^4 \Rightarrow dx = 4t^3 dt$$

$$\int \frac{t^2 \cdot 4t^3 dt}{(t^3 + 1)}$$

$$4 \int \frac{t^2(t^3 + 1 - 1)}{(t^3 + 1)} dt$$

$$4 \int t^2 dt - 4 \int \frac{t^2}{t^3 + 1} dt$$

$$\frac{4t^3}{3} - \frac{4}{3} \ln(t^3 + 1) + C$$

$$yx^{-1/2} = \frac{4x^{3/4}}{3} - \frac{4}{3} \ln(x^{3/4} + 1) + C$$

$$1 - \frac{4}{3} \log_e 2 = \frac{4}{3} - \frac{4}{3} \log_e 2 + C$$

$$\Rightarrow C = -\frac{1}{3}$$

$$y = \frac{4}{3} x^{5/4} - \frac{4}{3} \sqrt{x} \ln(x^{3/4} + 1) - \frac{\sqrt{x}}{3}$$

$$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4 \ln 9 - \frac{4}{3}$$

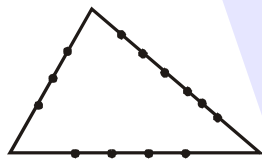
$$= \frac{124}{3} - \frac{32}{3} \ln 3 = 4 \left( \frac{31}{3} - \frac{8}{3} \ln 3 \right)$$

11. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to :

- (1) 364 (2) 240 (3) 333 (4) 360

Official Ans. by NTA (3)

Sol.



$$\begin{aligned} \text{Total Number of triangles formed} \\ &= {}^{14}C_3 - {}^3C_3 - {}^5C_3 - {}^6C_3 \\ &= 333 \end{aligned}$$

Option (3)

12. If  $x, y, z$  are in arithmetic progression with common difference  $d$ ,  $x \neq 3d$ , and the

determinant of the matrix  $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$  is zero,

then the value of  $k^2$  is

- (1) 72 (2) 12 (3) 36 (4) 6

Official Ans. by NTA (1)

Sol.  $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$$R_2 \rightarrow R_1 + R_3 - 2R_2$$

$$\Rightarrow \begin{vmatrix} 3 & 4\sqrt{2} & x \\ 0 & k - 6\sqrt{2} & 0 \\ 5 & k & z \end{vmatrix} = 0$$

$$\Rightarrow (k - 6\sqrt{2})(3z - 5x) = 0$$

$$\text{if } 3z - 5x = 0 \Rightarrow 3(x + 2d) - 5x = 0$$

$$\Rightarrow x = 3d \text{ (Not possible)}$$

$$\Rightarrow k = 6\sqrt{2} \Rightarrow k^2 = 72 \quad \text{Option (1)}$$

13. Let  $O$  be the origin. Let  $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$  and  $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$ ,  $x, y \in \mathbb{R}$ ,  $x > 0$ , be such that

$|\overrightarrow{PQ}| = \sqrt{20}$  and the vector  $\overrightarrow{OP}$  is perpendicular

to  $\overrightarrow{OQ}$ . If  $\overrightarrow{OR} = 3\hat{i} + z\hat{j} - 7\hat{k}$ ,  $z \in \mathbb{R}$ , is coplanar with  $\overrightarrow{OP}$  and  $\overrightarrow{OQ}$ , then the value of  $x^2 + y^2 + z^2$  is equal to

- (1) 7 (2) 9 (3) 2 (4) 1

Official Ans. by NTA (2)

Sol.  $\overrightarrow{OP} \perp \overrightarrow{OQ}$

$$\Rightarrow -x + 2y - 3x = 0$$

$$\Rightarrow y = 2x \quad \dots(i)$$

$$|\overrightarrow{PQ}|^2 = 20$$

$$\Rightarrow (x + 1)^2 + (y - 2)^2 + (1 + 3x)^2 = 20$$

$$\Rightarrow x = 1$$

$\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$  are coplanar.

$$\Rightarrow \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\Rightarrow 1(-14 - 3z) - 2(7 - 9) - 1(-z - 6) = 0$$

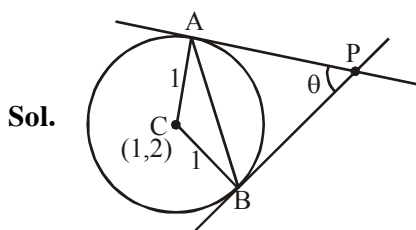
$$\Rightarrow z = -2$$

$$\therefore x^2 + y^2 + z^2 = 1 + 4 + 4 = 9 \quad \text{Option (2)}$$

14. Two tangents are drawn from a point P to the circle  $x^2 + y^2 - 2x - 4y + 4 = 0$ , such that the angle between these tangents is  $\tan^{-1}\left(\frac{12}{5}\right)$ , where  $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$ . If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of  $\Delta PAB$  and  $\Delta CAB$  is :

- (1) 11 : 4 (2) 9 : 4 (3) 3 : 1 (4) 2 : 1

Official Ans. by NTA (2)



Sol.

$$\tan \theta = \frac{12}{5}$$

$$PA = \cot \frac{\theta}{2}$$

$$\therefore \text{area of } \Delta PAB = \frac{1}{2}(PA)^2 \sin \theta = \frac{1}{2} \cot^2 \frac{\theta}{2} \sin \theta$$

$$= \frac{1}{2} \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right) \sin \theta$$

$$= \frac{1}{2} \left( \frac{1 + \frac{5}{13}}{1 - \frac{5}{13}} \right) \left( \frac{12}{13} \right) = \frac{1}{2} \cdot \frac{18}{8} \times \frac{2}{13} = \frac{27}{26}$$

$$\text{area of } \Delta CAB = \frac{1}{2} \sin \theta = \frac{1}{2} \left( \frac{12}{13} \right) = \frac{6}{13}$$

$$\therefore \frac{\text{area of } \Delta PAB}{\text{area of } \Delta CAB} = \frac{9}{4} \quad \text{Option (2)}$$

15. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} \left( 2 - \sin\left(\frac{1}{x}\right) \right) |x|, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ Then } f \text{ is :}$$

- (1) monotonic on  $(-\infty, 0) \cup (0, \infty)$   
 (2) not monotonic on  $(-\infty, 0)$  and  $(0, \infty)$   
 (3) monotonic on  $(0, \infty)$  only  
 (4) monotonic on  $(-\infty, 0)$  only

Official Ans. by NTA (2)

Sol. 
$$f(x) = \begin{cases} -x \left( 2 - \sin\left(\frac{1}{x}\right) \right) & x < 0 \\ 0 & x = 0 \\ x \left( 2 - \sin\left(\frac{1}{x}\right) \right) & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -\left( 2 - \sin\frac{1}{x} \right) - x \left( -\cos\frac{1}{x} \cdot \left( -\frac{1}{x^2} \right) \right) & x < 0 \\ \left( 2 - \sin\frac{1}{x} \right) + x \left( -\cos\frac{1}{x} \cdot \left( -\frac{1}{x^2} \right) \right) & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -2 + \sin\frac{1}{x} - \frac{1}{x} \cos\frac{1}{x} & x < 0 \\ 2 - \sin\frac{1}{x} + \frac{1}{x} \cos\frac{1}{x} & x > 0 \end{cases}$$

$f'(x)$  is an oscillating function which is non-monotonic in  $(-\infty, 0) \cup (0, \infty)$ .

Option (2)

16. Let L be a tangent line to the parabola  $y^2 = 4x - 20$  at  $(6, 2)$ . If L is also a tangent to the ellipse

$$\frac{x^2}{2} + \frac{y^2}{b} = 1, \text{ then the value of } b \text{ is equal to :}$$

- (1) 11 (2) 14 (3) 16 (4) 20

Official Ans. by NTA (2)

Sol. Tangent to parabola

$$2y = 2(x + 6) - 20$$

$$\Rightarrow y = x - 4$$

Condition of tangency for ellipse.

$$16 = 2(1)^2 + b$$

$$\Rightarrow b = 14$$

Option (2)

17. The value of the limit  $\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$  is equal to :

- (1)  $-\frac{1}{2}$  (2)  $-\frac{1}{4}$  (3) 0 (4)  $\frac{1}{4}$

Official Ans. by NTA (1)

Sol. 
$$\lim_{\theta \rightarrow 0} \frac{\tan(\pi(1 - \sin^2 \theta))}{\sin(2\pi \sin^2 \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{-\tan(\pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)}$$

$$= \lim_{\theta \rightarrow 0} - \left( \frac{\tan(\pi \sin^2 \theta)}{\pi \sin^2 \theta} \right) \left( \frac{2\pi \sin^2 \theta}{\sin(2\pi \sin^2 \theta)} \right) \times \frac{1}{2}$$

$$= -\frac{1}{2}$$

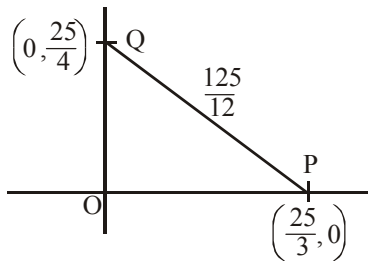
Option (1)

18. Let the tangent to the circle  $x^2 + y^2 = 25$  at the point  $R(3, 4)$  meet  $x$ -axis and  $y$ -axis at point  $P$  and  $Q$ , respectively. If  $r$  is the radius of the circle passing through the origin  $O$  and having centre at the incentre of the triangle  $OPQ$ , then  $r^2$  is equal to

- (1)  $\frac{529}{64}$  (2)  $\frac{125}{72}$  (3)  $\frac{625}{72}$  (4)  $\frac{585}{66}$

Official Ans. by NTA (3)

Sol. Tangent to circle  $3x + 4y = 25$



$$OP + OQ + OR = 25$$

$$\text{Incentre} = \left( \frac{\frac{25}{4} \times \frac{25}{3} + \frac{25}{4} \times \frac{25}{3}}{\frac{25}{4} + \frac{25}{3}}, \frac{\frac{25}{4} \times \frac{25}{3} + \frac{25}{4} \times \frac{25}{3}}{\frac{25}{4} + \frac{25}{3}} \right)$$

$$= \left( \frac{25}{12}, \frac{25}{12} \right)$$

$$\therefore r^2 = 2 \left( \frac{25}{12} \right)^2 = 2 \times \frac{625}{144} = \frac{625}{72}$$

Option (3)

19. If the Boolean expression  $(p \wedge q) \otimes (p \otimes q)$  is a tautology, then  $\otimes$  and  $\otimes$  are respectively given by

- (1)  $\rightarrow, \rightarrow$  (2)  $\wedge, \vee$  (3)  $\vee, \rightarrow$  (4)  $\wedge, \rightarrow$

Official Ans. by NTA (1)

Sol. Option (1)

$$(p \wedge q) \rightarrow (p \rightarrow q)$$

$$= \sim (p \wedge q) \vee (\sim p \vee q)$$

$$= (\sim p \vee \sim q) \vee (\sim p \vee q)$$

$$= \sim p \vee (\sim q \vee q)$$

$$= \sim p \vee t$$

$$= t$$

Option (2)

$$(p \wedge q) \wedge (p \vee q) = (p \wedge q) \text{ (Not a tautology)}$$

Option (3)

$$(p \wedge q) \vee (p \rightarrow q)$$

$$= (p \wedge q) \vee (\sim p \vee q)$$

$$= \sim p \vee q \text{ (Not a tautology)}$$

Option (4)

$$= (p \wedge q) \wedge (\sim p \vee q)$$

$$= p \wedge q \text{ (Not a tautology)}$$

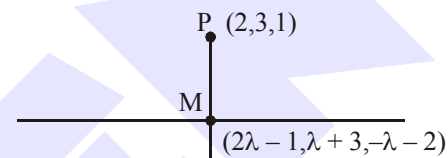
Option (1)

20. If the equation of plane passing through the mirror image of a point  $(2, 3, 1)$  with respect to line  $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1}$  and containing the line  $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1}$  is  $\alpha x + \beta y + \gamma z = 24$ , then  $\alpha + \beta + \gamma$  is equal to :

- (1) 20 (2) 19 (3) 18 (4) 21

Official Ans. by NTA (2)

Sol. Line  $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1}$



$$\vec{PM} = (2\lambda - 3, \lambda, -\lambda - 3)$$

$$\vec{PM} \perp (2\hat{i} + \hat{j} - \hat{k})$$

$$4\lambda - 6 + \lambda + \lambda + 3 = 0 \Rightarrow \lambda = \frac{1}{2}$$

$$\therefore M \equiv \left( 0, \frac{7}{2}, \frac{-5}{2} \right)$$

$\therefore$  Reflection  $(-2, 4, -6)$

$$\text{Plane : } \begin{vmatrix} x-2 & y-1 & z+1 \\ 3 & -2 & 1 \\ 4 & -3 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(-10+3) - (y-1)(15-4) + (z+1)(-1) = 0$$

$$\Rightarrow -7x + 14 - 11y + 11 - z - 1 = 0$$

$$\Rightarrow 7x + 11y + z = 24$$

$$\therefore \alpha = 7, \beta = 11, \gamma = 1$$

$$\alpha + \beta + \gamma = 19$$

Option (2)

### SECTION-B

1. If  $1, \log_{10}(4^x - 2)$  and  $\log_{10}\left(4^x + \frac{18}{5}\right)$  are in arithmetic progression for a real number  $x$ , then the value of the determinant

$$\begin{vmatrix} 2\left(x - \frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$$
 is equal to :

Official Ans. by NTA (2)

**Sol.**  $2\log_{10}(4^x - 2) = 1 + \log_{10}\left(4^x + \frac{18}{5}\right)$

$$(4^x - 2)^2 = 10\left(4^x + \frac{18}{5}\right)$$

$$(4^x)^2 + 4 - 4(4^x) - 32 = 0$$

$$(4^x - 16)(4^x + 2) = 0$$

$$4^x = 16$$

$$x = 2$$

$$\begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 3(-2) - 1(0 - 4) + 4(1)$$

$$= -6 + 4 + 4 = 2$$

2. Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be defined as  $f(x) = ax^2 + bx + c$  for all  $x \in [-1, 1]$ , where  $a, b, c \in \mathbb{R}$  such that  $f(-1) = 2$ ,  $f'(-1) = 1$  and for  $x \in (-1, 1)$  the maximum value of  $f''(x)$  is  $\frac{1}{2}$ . If  $f(x) \leq \alpha$ ,  $x \in [-1, 1]$ , then the least value of  $\alpha$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (5)**

**Sol.**  $f : [-1, 1] \rightarrow \mathbb{R}$

$$f(x) = ax^2 + bx + c$$

$$f(-1) = a - b + c = 2 \quad \dots(1)$$

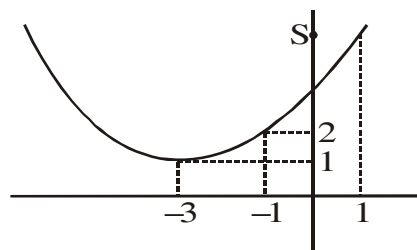
$$f'(-1) = -2a + b = 1 \quad \dots(2)$$

$$f''(x) = 2a$$

$$\Rightarrow \text{Max. value of } f''(x) = 2a = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{4}; \quad b = \frac{3}{2}; \quad c = \frac{13}{4}$$

$$\therefore f(x) = \frac{x^2}{4} + \frac{3}{2}x + \frac{13}{4}$$



$$\text{For, } x \in [-1, 1] \Rightarrow 2 \leq f(x) \leq 5$$

$$\therefore \text{Least value of } \alpha \text{ is } 5$$

3. Let  $f : [-3, 1] \rightarrow \mathbb{R}$  be given as

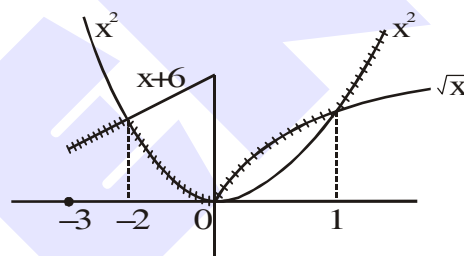
$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \leq x \leq 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \leq x \leq 1. \end{cases}$$

If the area bounded by  $y = f(x)$  and  $x$ -axis is  $A$ , then the value of  $6A$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (41)**

**Sol.**  $f : [-3, 1] \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \leq x \leq 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \leq x \leq 1 \end{cases}$$



area bounded by  $y = f(x)$  and  $x$ -axis

$$= \int_{-3}^{-2} (x+6) dx + \int_{-2}^0 x^2 dx + \int_0^1 \sqrt{x} dx$$

$$A = \frac{41}{6}$$

$$6A = 41$$

4. Let  $\tan \alpha, \tan \beta$  and  $\tan \gamma$ ;  $\alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}$ ,

$n \in \mathbb{N}$  be the slopes of three line segments  $OA$ ,  $OB$  and  $OC$ , respectively, where  $O$  is origin. If circumcentre of  $\Delta ABC$  coincides with origin and its orthocentre lies on  $y$ -axis, then the value

$$\text{of } \left( \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma} \right)^2 \text{ is equal to :}$$

**Official Ans. by NTA (144)**



**Sol.** Since orthocentre and circumcentre both lies on y-axis

$\Rightarrow$  Centroid also lies on y-axis

$$\Rightarrow \Sigma \cos \alpha = 0$$

$$\cos \alpha + \cos \beta + \cos \gamma = 0$$

$$\Rightarrow \cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma = 3 \cos \alpha \cos \beta \cos \gamma$$

$$\therefore \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma}$$

$$= \frac{4(\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3(\cos \alpha + \cos \beta + \cos \gamma)}{\cos \alpha \cos \beta \cos \gamma}$$

$$= 12$$

**5.** Consider a set of  $3n$  numbers having variance 4. In this set, the mean of first  $2n$  numbers is 6 and the mean of the remaining  $n$  numbers is 3. A new set is constructed by adding 1 into each of first  $2n$  numbers, and subtracting 1 from each of the remaining  $n$  numbers. If the variance of the new set is  $k$ , then  $9k$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (68)**

**Sol.** Let number be  $a_1, a_2, a_3, \dots, a_{2n}, b_1, b_2, b_3, \dots, b_n$

$$\sigma^2 = \frac{\sum a^2 + \sum b^2}{3n} - (5)^2$$

$$\Rightarrow \sum a^2 + \sum b^2 = 87n$$

Now, distribution becomes

$$a_1 + 1, a_2 + 1, a_3 + 1, \dots, a_{2n} + 1, b_1 - 1, b_2 - 1, \dots, b_n - 1$$

Variance

$$= \frac{\sum (a+1)^2 + \sum (b-1)^2}{3n} - \left( \frac{12n+2n+3n-n}{3n} \right)^2$$

$$= \frac{(\sum a^2 + 2n + 2\sum a) + (\sum b^2 + n - 2\sum b)}{3n}$$

$$= \frac{(\sum a^2 + 2n + 2\sum a) + (\sum b^2 + n - 2\sum b)}{3n} - \left( \frac{16}{3} \right)^2$$

$$= \frac{87n + 3n + 2(12n) - 2(3n)}{3n} - \left( \frac{16}{3} \right)^2$$

$$\Rightarrow k = \frac{108}{3} - \left( \frac{16}{3} \right)^2$$

$$\Rightarrow 9k = 3(108) - (16)^2 = 324 - 256 = 68$$

**Ans. 68.00**

**6.** Let the coefficients of third, fourth and fifth terms in the expansion of  $\left(x + \frac{a}{x^2}\right)^n$ ,  $x \neq 0$ , be in the ratio  $12 : 8 : 3$ . Then the term independent of  $x$  in the expansion, is equal to \_\_\_\_\_.

**Official Ans. by NTA (4)**

$$\text{Sol. } T_{r+1} = {}^n C_r (x)^{n-r} \left(\frac{a}{x^2}\right)^r$$

$$= {}^n C_r a^r x^{n-3r}$$

$${}^n C_2 a^2 : {}^n C_3 a^3 : {}^n C_4 a^4 = 12 : 8 : 3$$

After solving

$$n = 6, a = \frac{1}{2}$$

For term independent of 'x'  $\Rightarrow n = 3r$   
 $r = 2$

$$\therefore \text{Coefficient is } {}^6 C_2 \left(\frac{1}{2}\right)^2 = \frac{15}{4}$$

Nearest integer is 4.

**Ans. 4**

**7.** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  such that

$AB = B$  and  $a + d = 2021$ , then the value of  $ad - bc$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (2020)**

$$\text{Sol. } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$AB = B$$

$$\Rightarrow (A - I) B = O$$

$$\Rightarrow |A - I| = 0, \text{ since } B \neq O$$

$$\begin{vmatrix} (a-1) & b \\ c & (d-1) \end{vmatrix} = 0$$

$$ad - bc = 2020$$

**8.** Let  $\vec{x}$  be a vector in the plane containing vectors  $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$  and  $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ . If the vector  $\vec{x}$  is perpendicular to  $(3\hat{i} + 2\hat{j} - \hat{k})$  and

its projection on  $\vec{a}$  is  $\frac{17\sqrt{6}}{2}$ , then the value of

$|\vec{x}|^2$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (486)**

**Sol.** Let  $\vec{x} = \lambda\vec{a} + \mu\vec{b}$  ( $\lambda$  and  $\mu$  are scalars)

$$\vec{x} = \hat{i}(2\lambda + \mu) + \hat{j}(2\mu - \lambda) + \hat{k}(\lambda - \mu)$$

$$\text{Since } \vec{x} \cdot (3\hat{i} + 2\hat{j} - \hat{k}) = 0$$

$$3\lambda + 8\mu = 0 \quad \dots(1)$$

Also Projection of  $\vec{x}$  on  $\vec{a}$  is  $\frac{17\sqrt{6}}{2}$

$$\frac{\vec{x} \cdot \vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$$

$$6\lambda - \mu = 51 \quad \dots(2)$$

From (1) and (2)

$$\lambda = 8, \mu = -3$$

$$\vec{x} = 13\hat{i} - 14\hat{j} + 11\hat{k}$$

$$|\vec{x}|^2 = 486 \quad \text{Ans.}$$

**9.** Let  $I_n = \int_1^e x^{19} (\log|x|)^n dx$ , where  $n \in \mathbb{N}$ . If

$(20)I_{10} = \alpha I_9 + \beta I_8$ , for natural numbers  $\alpha$  and  $\beta$ , then  $\alpha - \beta$  equal to \_\_\_\_\_.

**Official Ans. by NTA (1)**

**Sol.**  $I_n = \int_1^e x^{19} (\log|x|)^n dx$

$$I_n = \left[ (\log|x|)^{19} \frac{x^{20}}{20} \right]_1^e - \int_1^e n(\log|x|)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{20}}{20} dx$$

$$20I_n = e^{20} - nI_{n-1}$$

$$\therefore 20I_{10} = e^{20} - 10I_9$$

$$20I_9 = e^{20} - 9I_8$$

$$\Rightarrow 20I_{10} = 10I_9 + 9I_8$$

$$\alpha = 10, \beta = 9$$

**10.** Let P be an arbitrary point having sum of the squares of the distance from the planes  $x + y + z = 0$ ,  $lx - nz = 0$  and  $x - 2y + z = 0$ , equal to 9. If the locus of the point P is  $x^2 + y^2 + z^2 = 9$ , then the value of  $l - n$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (0)**

**Sol.** Let point P is  $(\alpha, \beta, \gamma)$

$$\left( \frac{\alpha + \beta + \gamma}{\sqrt{3}} \right)^2 + \left( \frac{l\alpha - n\gamma}{\sqrt{l^2 + n^2}} \right)^2 + \left( \frac{\alpha - 2\beta + \gamma}{\sqrt{6}} \right)^2 = 9$$

Locus is

$$\frac{(x + y + z)^2}{3} + \frac{(lx - nz)^2}{l^2 + n^2} + \frac{(x - 2y + z)^2}{6} = 9$$

$$x^2 \left( \frac{1}{2} + \frac{l^2}{l^2 + n^2} \right) + y^2 + z^2 \left( \frac{1}{2} + \frac{n^2}{l^2 + n^2} \right) + 2zx \left( \frac{1}{2} - \frac{ln}{l^2 + n^2} \right) - 9 = 0$$

Since its given that  $x^2 + y^2 + z^2 = 9$

After solving  $l = n$