

FINAL JEE-MAIN EXAMINATION – MARCH, 2021

(Held On Tuesday 16th March, 2021) TIME : 9 : 00 AM to 12 : 00 NOON

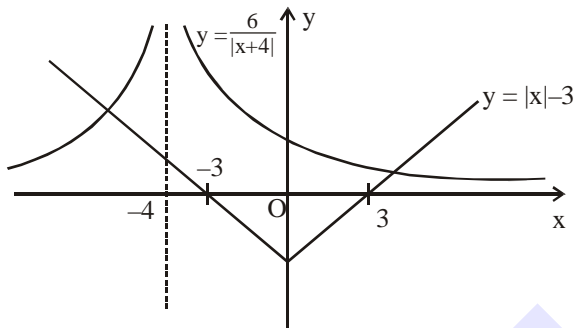
MATHEMATICS

SECTION-A

1. The number of elements in the set $\{x \in \mathbb{R} : (|x| - 3)|x + 4| = 6\}$ is equal to
 (1) 3 (2) 2 (3) 4 (4) 1

Official Ans. by NTA (2)

Sol. $x \neq -4$
 $(|x| - 3)(|x + 4|) = 6$
 $\Rightarrow |x| - 3 = \frac{6}{|x+4|}$

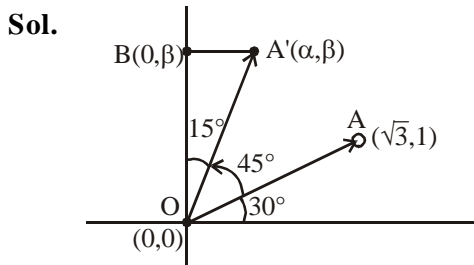


No. of solutions = 2

2. Let a vector $\alpha \hat{i} + \beta \hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i} + \hat{j}$ by an angle 45° about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices (α, β) , $(0, \beta)$ and $(0, 0)$ is equal to

- (1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{\sqrt{2}}$ (4) $2\sqrt{2}$

Official Ans. by NTA (1)



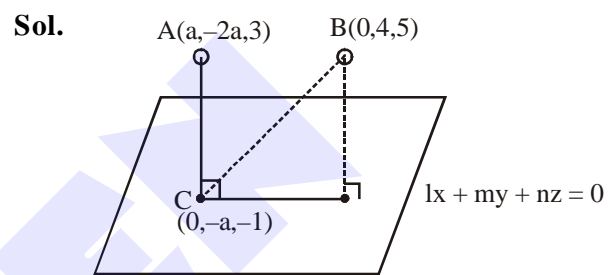
Area of $\Delta(OA'B) = \frac{1}{2} OA' \cos 15^\circ \times OA' \sin 15^\circ$
 $= \frac{1}{2} (OA')^2 \frac{\sin 30^\circ}{2}$
 $= (3+1) \times \frac{1}{8} = \frac{1}{2}$

TEST PAPER WITH SOLUTION

3. If for $a > 0$, the feet of perpendiculars from the points $A(a, -2a, 3)$ and $B(0, 4, 5)$ on the plane $lx + my + nz = 0$ are points $C(0, -a, -1)$ and D respectively, then the length of line segment CD is equal to :

- (1) $\sqrt{31}$ (2) $\sqrt{41}$
 (3) $\sqrt{55}$ (4) $\sqrt{66}$

Official Ans. by NTA (4)



C lies on plane $\Rightarrow -ma - n = 0 \Rightarrow \frac{m}{n} = -\frac{1}{a} \dots(1)$

$\vec{CA} \parallel l\hat{i} + m\hat{j} + n\hat{k}$

$\frac{a-0}{l} = \frac{-a}{m} = \frac{4}{n} \Rightarrow \frac{m}{n} = -\frac{a}{4} \dots(2)$

From (1) & (2)

$-\frac{1}{a} = \frac{-a}{4} \Rightarrow a^2 = 4 \Rightarrow a = 2$ (since $a > 0$)

From (2) $\frac{m}{n} = \frac{-1}{2}$

Let $m = -t \Rightarrow n = 2t$

$\frac{2}{l} = \frac{-2}{-t} \Rightarrow l = t$

So plane : $t(x - y + 2z) = 0$

$BD = \frac{6}{\sqrt{6}} = \sqrt{6} \quad C \equiv (0, -2, -1)$

$CD = \sqrt{BC^2 - BD^2}$
 $= \sqrt{(0^2 + 6^2 + 6^2) - (\sqrt{6})^2}$
 $= \sqrt{66}$

4. Consider three observations a, b and c such that $b = a + c$. If the standard deviation of $a + 2$, $b + 2$, $c + 2$ is d, then which of the following is true ?

- (1) $b^2 = 3(a^2 + c^2) + 9d^2$
- (2) $b^2 = a^2 + c^2 + 3d^2$
- (3) $b^2 = 3(a^2 + c^2 + d^2)$
- (4) $b^2 = 3(a^2 + c^2) - 9d^2$

Official Ans. by NTA (4)

Sol. For a, b, c

$$\text{mean} = \frac{a+b+c}{3} (= \bar{x})$$

$$b = a + c$$

$$\Rightarrow \bar{x} = \frac{2b}{3} \quad \dots\dots(1)$$

$$\text{S.D. } (a + 2, b + 2, c + 2) = \text{S.D. } (a, b, c) = d$$

$$\Rightarrow d^2 = \frac{a^2 + b^2 + c^2}{3} - (\bar{x})^2$$

$$\Rightarrow d^2 = \frac{a^2 + b^2 + c^2}{3} - \frac{4b^2}{9}$$

$$\Rightarrow 9d^2 = 3(a^2 + b^2 + c^2) - 4b^2$$

$$\Rightarrow b^2 = 3(a^2 + c^2) - 9d^2$$

5. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10} \sin x + \log_{10} \cos x = -1$

$$\text{and } \log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1), n > 0,$$

then the value of n is equal to :

- (1) 20
- (2) 12
- (3) 9
- (4) 16

Official Ans. by NTA (2)

Sol. $x \in \left(0, \frac{\pi}{2}\right)$

$$\log_{10} \sin x + \log_{10} \cos x = -1$$

$$\Rightarrow \log_{10} \sin x \cdot \cos x = -1$$

$$\Rightarrow \sin x \cdot \cos x = \frac{1}{10} \quad \dots\dots(1)$$

$$\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$$

$$\Rightarrow \sin x + \cos x = 10^{\left(\frac{\log_{10} \sqrt{n} - 1}{2}\right)} = \sqrt{\frac{n}{10}}$$

by squaring

$$1 + 2\sin x \cdot \cos x = \frac{n}{10}$$

$$\Rightarrow 1 + \frac{1}{5} = \frac{n}{10} \quad \Rightarrow n = 12$$

6. Let $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i = \sqrt{-1}$. Then, the system of

linear equations $A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has :

- (1) A unique solution
- (2) Infinitely many solutions
- (3) No solution
- (4) Exactly two solutions

Official Ans. by NTA (3)

Sol. $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^4 = 2^2 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 = 64 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow 128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow 128 \begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow x - y = \frac{1}{16} \quad \dots\dots(1)$$

$$\& \quad -x + y = \frac{1}{2} \quad \dots\dots(2)$$

\Rightarrow From (1) & (2) : No solution.

7. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point (a, 0) $a \neq 0$, then 'a' must be greater than :

- (1) $\frac{1}{2}$
- (2) $-\frac{1}{2}$
- (3) -1
- (4) 1

Official Ans. by NTA (4)

Sol. For standard parabola
For more than 3 normals (on axis)

$$x > \frac{L}{2} \quad (\text{where } L \text{ is length of L.R.})$$

For $y^2 = 2x$

L.R. = 2

for (a, 0)

$$a > \frac{\text{L.R.}}{2} \Rightarrow a > 1$$

8. Let the position vectors of two points P and Q be $3\hat{i} - \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} - 4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are (4, -1, 2) and (-2, 1, -2), respectively. Let lines PR and QS intersect at T. If the vector \overline{TA} is perpendicular to both \overline{PR} and \overline{QS} and the length of vector \overline{TA} is $\sqrt{5}$ units, then the modulus of a position vector of A is :

(1) $\sqrt{482}$ (2) $\sqrt{171}$

(3) $\sqrt{5}$ (4) $\sqrt{227}$

Official Ans. by NTA (2)

Sol. P(3, -1, 2)

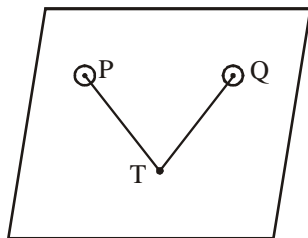
Q(1, 2, -4)

$$\overline{PR} \parallel 4\hat{i} - \hat{j} + 2\hat{k}$$

$$\overline{QS} \parallel -2\hat{i} + \hat{j} - 2\hat{k}$$

dr's of normal to the plane containing P, T & Q will be proportional to :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$$



$$\therefore \frac{l}{0} = \frac{m}{4} = \frac{n}{2}$$

For point, T : $\overline{PT} = \frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$

$$\overline{QT} = \frac{x-1}{-2} = \frac{y-2}{1} = \frac{z+4}{-2} = \mu$$

$$T : (4\lambda + 3, -\lambda - 1, 2\lambda + 2)$$

$$\cong (2\mu + 1, \mu + 2, -2\mu - 4)$$

$$4\lambda + 3 = -2\mu + 1 \Rightarrow 2\lambda + \mu = -1$$

$$\lambda + \mu = -3 \Rightarrow \lambda = 2$$

$$\& \mu = -5 \quad \lambda + \mu = -3 \Rightarrow \lambda = 2$$

So point T : (11, -3, 6)

$$\overline{OA} = (11\hat{i} - 3\hat{j} + 6\hat{k}) \pm \left(\frac{2\hat{j} + \hat{k}}{\sqrt{5}} \right) \sqrt{5}$$

$$\overline{OA} = (11\hat{i} - 3\hat{j} + 6\hat{k}) \pm (2\hat{j} + \hat{k})$$

$$\overline{OA} = 11\hat{i} - \hat{j} + 7\hat{k}$$

or

$$9\hat{i} - 5\hat{j} + 5\hat{k}$$

$$|\overline{OA}| = \sqrt{121 + 1 + 49} = \sqrt{171}$$

or

$$\sqrt{81 + 25 + 25} = \sqrt{131}$$

9. Let the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as :

$$f(x) = \begin{cases} x+2, & x < 0 \\ x^2, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x^3, & x < 1 \\ 3x-2, & x \geq 1 \end{cases}$$

Then, the number of points in \mathbb{R} where $(f \circ g)(x)$ is NOT differentiable is equal to :

(1) 3 (2) 1

(3) 0 (4) 2

Official Ans. by NTA (2)

Sol. $f(g(x)) = \begin{cases} g(x)+2, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$

$$= \begin{cases} x^3 + 2, & x < 0 \\ x^6, & x \in [0, 1) \\ (3x-2)^2, & x \in [1, \infty) \end{cases}$$

$$(f \circ g(x))' = \begin{cases} 3x^2, & x < 0 \\ 6x^5, & x \in (0, 1) \\ 2(3x-2) \times 3, & x \in (1, \infty) \end{cases}$$

At '0'

L.H.L. \neq R.H.L. (Discontinuous)

At '1'

L.H.D. = 6 = R.H.D.

$\Rightarrow f \circ g(x)$ is differentiable for $x \in \mathbb{R} - \{0\}$

10. Which of the following Boolean expression is a tautology ?

- (1) $(p \wedge q) \vee (p \vee q)$
- (2) $(p \wedge q) \vee (p \rightarrow q)$
- (3) $(p \wedge q) \wedge (p \rightarrow q)$
- (4) $(p \wedge q) \rightarrow (p \rightarrow q)$

Official Ans. by NTA (4)

Sol.

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

$(p \wedge q) \rightarrow (p \rightarrow q)$ is tautology

11. Let a complex number z , $|z| \neq 1$,

satisfy $\log_{\frac{1}{\sqrt{2}}} \left(\frac{|z|+11}{(|z|-1)^2} \right) \leq 2$. Then, the largest

value of $|z|$ is equal to _____ .

- (1) 8
- (2) 7
- (3) 6
- (4) 5

Official Ans. by NTA (2)

Sol. $\log_{\frac{1}{\sqrt{2}}} \left(\frac{|z|+11}{(|z|-1)^2} \right) \leq 2$

$$\frac{|z|+11}{(|z|-1)^2} \geq \frac{1}{2}$$

$$2|z| + 22 \geq (|z| - 1)^2$$

$$2|z| + 22 \geq |z|^2 + 1 - 2|z|$$

$$|z|^2 - 4|z| - 21 \leq 0$$

$$\Rightarrow |z| \leq 7$$

\therefore Largest value of $|z|$ is 7

12. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then $(n - 1)$ is divisible by :

- (1) 26
- (2) 30
- (3) 8
- (4) 7

Official Ans. by NTA (1)

Sol. $(3^{1/4} + 5^{1/8})^{60}$
 ${}^{60}C_r (3^{1/4})^{60-r} \cdot (5^{1/8})^r$

$${}^{60}C_r (3)^{\frac{60-r}{4}} \cdot 5^{\frac{r}{8}}$$

For rational terms.

$$\frac{r}{8} = k ; \quad 0 \leq r \leq 60$$

$$0 \leq 8k \leq 60$$

$$0 \leq k \leq \frac{60}{8}$$

$$0 \leq k \leq 7.5$$

$k = 0, 1, 2, 3, 4, 5, 6, 7$

$\frac{60-8k}{4}$ is always divisible by 4 for all value of k .

Total rational terms = 8

Total terms = 61

irrational terms = 53

$$n - 1 = 53 - 1 = 52$$

52 is divisible by 26.

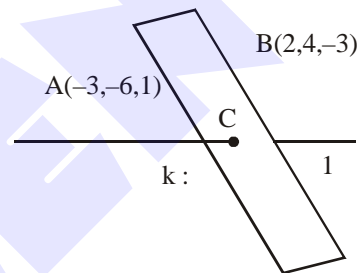
13. Let P be a plane $lx + my + nz = 0$ containing

the line, $\frac{1-x}{1} = \frac{y+4}{2} = \frac{z+2}{3}$. If plane P divides the line segment AB joining points $A(-3, -6, 1)$ and $B(2, 4, -3)$ in ratio $k : 1$ then the value of k is equal to :

- (1) 1.5
- (2) 3
- (3) 2
- (4) 4

Official Ans. by NTA (3)

Sol.



Point C is

$$\left(\frac{2k-3}{k+1}, \frac{4k-6}{k+1}, \frac{-3k+1}{k+1} \right)$$

$$\frac{x-1}{-1} = \frac{y+4}{2} = \frac{z+2}{3}$$

Plane $lx + my + nz = 0$

$$l(-1) + m(2) + n(3) = 0$$

$$-l + 2m + 3n = 0 \quad \dots\dots(1)$$

It also satisfy point $(1, -4, -2)$

$$l - 4m - 2n = 0 \quad \dots\dots(2)$$

Solving (1) and (2)

$$2m + 3n = 4m + 2n$$

$$n = 2m$$

$$l - 4m - 4m = 0$$

$$l = 8m$$

$$\frac{l}{8} = \frac{m}{1} = \frac{n}{2}$$

$$l : m : n = 8 : 1 : 2$$

Plane is $8x + y + 2z = 0$

It will satisfy point C

$$8 \left(\frac{2k-3}{k+1} \right) + \left(\frac{4k-6}{k+1} \right) + 2 \left(\frac{-3k+1}{k+1} \right) = 0$$

$$16k - 24 + 4k - 6 - 6k + 2 = 0$$

$$14k = 28 \quad \therefore k = 2$$

14. The range of $a \in \mathbb{R}$ for which the function

$$f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7) \cot\left(\frac{x}{2}\right) \sin^2\left(\frac{x}{2}\right),$$

$x \neq 2n\pi, n \in \mathbb{N}$, has critical points, is :

- (1) $(-3, 1)$ (2) $\left[-\frac{4}{3}, 2\right]$
 (3) $[1, \infty)$ (4) $(-\infty, -1]$

Official Ans. by NTA (2)

Sol. $f(x) = (4a - 3)(x + \log_e 5) + (a - 7) \sin x$

$$f(x) = (4a - 3)(1) + (a - 7) \cos x = 0$$

$$\Rightarrow \cos x = \frac{3 - 4a}{a - 7}$$

$$-1 \leq \frac{3 - 4a}{a - 7} < 1$$

$$\frac{3 - 4a}{a - 7} + 1 \geq 0$$

$$\frac{3 - 4a + a - 7}{a - 7} \geq 0$$

$$\frac{-3a - 4}{a - 7} \geq 0$$

$$\frac{3a + 4}{a - 7} \leq 0$$

$$\frac{3 - 4a}{a - 7} < 1$$

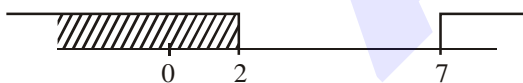
$$\frac{3 - 4a}{a - 7} - 1 < 0$$

$$\frac{3 - 4a - a + 7}{a - 7} < 0$$

$$\frac{-5a + 10}{a - 7} < 0$$

$$\frac{5a - 10}{a - 7} > 0$$

$$\frac{5(a - 2)}{a - 7} > 0$$



$$a \in \left[-\frac{4}{3}, 2\right)$$

Check end point $\left[-\frac{4}{3}, 2\right)$

15. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

- (1) $\frac{3}{4}$ (2) $\frac{52}{867}$
 (3) $\frac{39}{50}$ (4) $\frac{22}{425}$

Official Ans. by NTA (3)

Sol. E_1 : Event denotes spade is missing

$$P(E_1) = \frac{1}{4}; P(\bar{E}_1) = \frac{3}{4}$$

A : Event drawn two cards are spade

$$P(A) = \frac{\frac{1}{4} \times \binom{12}{51} C_2 + \frac{3}{4} \times \binom{13}{51} C_2 + \frac{3}{4} \times \binom{13}{51} C_2}{\frac{1}{4} \times \binom{12}{51} C_2 + \frac{3}{4} \times \binom{13}{51} C_2}$$

$$= \frac{39}{50}$$

16. Let $[x]$ denote greatest integer less than or

equal to x . If for $n \in \mathbb{N}$, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$,

then $\sum_{j=0}^{\lfloor \frac{3n}{2} \rfloor} a_{2j} + 4 \sum_{j=0}^{\lfloor \frac{3n-1}{2} \rfloor} a_{2j+1} + 1$ is equal to :

- (1) 2 (2) 2^{n-1}
 (3) 1 (4) n

Official Ans. by NTA (3)

Sol. $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$

$$(1 - x + x^3)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_{3n} x^{3n}$$

$$\sum_{j=0}^{\lfloor \frac{3n}{2} \rfloor} a_{2j} = \text{Sum of } a_0 + a_2 + a_4 + \dots$$

$$\sum_{j=0}^{\lfloor \frac{3n-1}{2} \rfloor} a_{2j+1} + 1 = \text{Sum of } a_1 + a_3 + a_5 + \dots$$

put $x = 1$

$$1 = a_0 + a_1 + a_2 + a_3 + \dots + a_{3n} \quad \dots (A)$$

Put $x = -1$

$$1 = a_0 - a_1 + a_2 - a_3 + \dots + (-1)^{3n} a_{3n} \quad \dots (B)$$

Solving (A) and (B)

$$a_0 + a_2 + a_4 + \dots = 1$$

$$a_1 + a_3 + a_5 + \dots = 0$$

$$\sum_{j=0}^{\lfloor \frac{3n}{2} \rfloor} a_{2j} + 4 \sum_{j=0}^{\lfloor \frac{3n-1}{2} \rfloor} a_{2j+1} = 1$$

17. If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y \tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over \mathbb{R} is equal to :

- (1) 8 (2) $\frac{1}{2}$ (3) $-\frac{15}{4}$ (4) $\frac{1}{8}$

Official Ans. by NTA (4)

Sol. $\frac{dy}{dx} + 2y \tan x = \sin x$

I.F. = $e^{\int 2 \tan x dx} = e^{2 \ln \sec x}$

I.F. = $\sec^2 x$

$y \cdot (\sec^2 x) = \int \sin x \cdot \sec^2 x dx$

$y \cdot (\sec^2 x) = \int \sec x \tan x dx$

$y \cdot (\sec^2 x) = \sec x + C$

$x = \frac{\pi}{3}; y = 0$

$\Rightarrow C = -2$

$\Rightarrow y = \frac{\sec x - 2}{\sec^2 x} = \cos x - 2 \cos^2 x$

$y = t - 2t^2 \Rightarrow \frac{dy}{dt} = 1 - 4t = 0 \Rightarrow t = \frac{1}{4}$

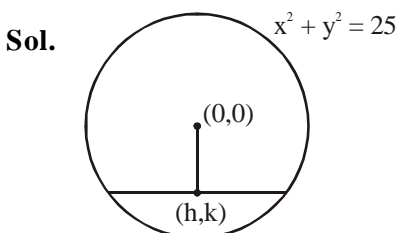
$\therefore \max = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$

18. The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the

hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is :

- (1) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$
 (2) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$
 (3) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$
 (4) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

Official Ans. by NTA (4)



Equation of chord

$y - k = -\frac{h}{k}(x - h)$

$ky - k^2 = -hx + h^2$
 $hx + ky = h^2 + k^2$

$y = -\frac{hx}{k} + \frac{h^2 + k^2}{k}$

tangent to $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$c^2 = a^2m^2 - b^2$

$\left(\frac{h^2 + k^2}{k}\right)^2 = 9\left(-\frac{h}{k}\right)^2 - 16$

$(x^2 + y^2)^2 = 9x^2 - 16y^2$

19. The number of roots of the equation,

$(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$

in the interval $[0, \pi]$ is equal to :

- (1) 3 (2) 4 (3) 8 (4) 2

Official Ans. by NTA (2)

Sol. $(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$

$(81)^{\sin^2 x} + \frac{(81)^1}{(18)^{\sin^2 x}} = 30$

$(81)^{\sin^2 x} = t$

$t + \frac{81}{t} = 30$

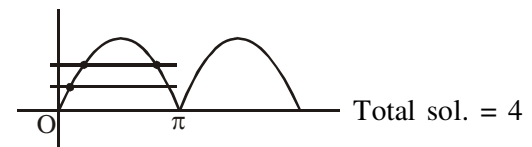
$t^2 - 30t + 81 = 0$

$(t - 3)(t - 27) = 0$

$(81)^{\sin^2 x} = 3^1$ or $(81)^{\sin^2 x} = 3^3$

$3^{4\sin^2 x} = 3^1$ or $3^{4\sin^2 x} = 3^3$

$\sin^2 x = \frac{1}{4}$ or $\sin^2 x = \frac{3}{4}$



20. Let $S_k = \sum_{r=1}^k \tan^{-1}\left(\frac{6^r}{2^{2r+1} + 3^{2r+1}}\right)$. Then $\lim_{k \rightarrow \infty} S_k$ is

equal to :

(1) $\tan^{-1}\left(\frac{3}{2}\right)$ (2) $\frac{\pi}{2}$

(3) $\cot^{-1}\left(\frac{3}{2}\right)$ (4) $\tan^{-1}(3)$

Official Ans. by NTA (3)

Sol. $S_k = \sum_{r=1}^k \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right)$

Divide by 3^{2r}

$$\sum_{r=1}^k \tan^{-1} \left(\frac{\left(\frac{2}{3}\right)^r}{\left(\frac{2}{3}\right)^{2r} \cdot 2 + 3} \right)$$

$$\sum_{r=1}^k \tan^{-1} \left(\frac{\left(\frac{2}{3}\right)^r}{3 \left(\left(\frac{2}{3}\right)^{2r+1} + 1 \right)} \right)$$

Let $\left(\frac{2}{3}\right)^r = t$

$$\sum_{r=1}^k \tan^{-1} \left(\frac{\frac{t}{3}}{1 + \frac{2}{3}t^2} \right)$$

$$\sum_{r=1}^k \tan^{-1} \left(\frac{t - \frac{2t}{3}}{1 + t \cdot \frac{2t}{3}} \right)$$

$$\sum_{r=1}^k \left(\tan^{-1}(t) - \tan^{-1}\left(\frac{2t}{3}\right) \right)$$

$$\sum_{r=1}^k \left(\tan^{-1}\left(\frac{2}{3}\right)^r - \tan^{-1}\left(\frac{2}{3}\right)^{r+1} \right)$$

$$S_k = \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{3}\right)^{k+1}$$

$$S_\infty = \lim_{k \rightarrow \infty} \left(\tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{3}\right)^{k+1} \right)$$

$$= \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}(0)$$

$$\therefore S_\infty = \tan^{-1}\left(\frac{2}{3}\right) = \cot^{-1}\left(\frac{3}{2}\right)$$

SECTION-B

1. Consider an arithmetic series and a geometric series having four initial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to _____ .

Official Ans. by NTA (3)

- Sol.** **GP :** 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192

AP : 11, 16, 21, 26, 31, 36

Common terms : 16, 256, 4096 only

2. Let $f : (0, 2) \rightarrow \mathbb{R}$ be defined as

$$f(x) = \log_2 \left(1 + \tan \left(\frac{\pi x}{4} \right) \right).$$

Then, $\lim_{n \rightarrow \infty} \frac{2}{n} \left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f(1) \right)$ is equal to _____ .

Official Ans. by NTA (1)

Sol. $E = 2 \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} f\left(\frac{r}{n}\right)$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi x}{4} \right) dx \quad \dots(i)$$

replacing $x \rightarrow 1 - x$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi}{4} (1 - x) \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \left(\frac{\pi}{4} - \frac{\pi}{4} x \right) \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \frac{1 + \tan \frac{\pi}{4} x}{1 + \tan \frac{\pi}{4} x} \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(\frac{2}{1 + \tan \frac{\pi x}{4}} \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \left(\ln 2 - \ln \left(1 + \tan \frac{\pi x}{4} \right) \right) dx \quad \dots(ii)$$

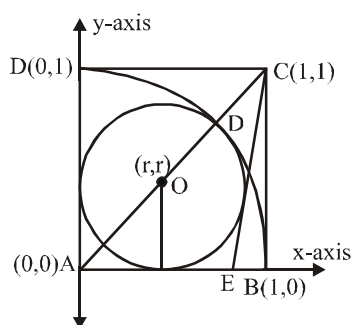
equation (i) + (ii)

$$E = 1$$

3. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to _____.

Official Ans. by NTA (1)

Sol.



Here $AO + OD = 1$ or $(\sqrt{2} + 1)r = 1$

$$\Rightarrow r = \sqrt{2} - 1$$

equation of circle $(x - r)^2 + (y - r)^2 = r^2$

Equation of CE

$$y - 1 = m(x - 1)$$

$$mx - y + 1 - m = 0$$

It is tangent to circle

$$\therefore \left| \frac{mr - r + 1 - m}{\sqrt{m^2 + 1}} \right| = r$$

$$\left| \frac{(m-1)r + 1 - m}{\sqrt{m^2 + 1}} \right| = r$$

$$\frac{(m-1)^2 (r-1)^2}{m^2 + 1} = r^2$$

Put $r = \sqrt{2} - 1$

On solving $m = 2 - \sqrt{3}, 2 + \sqrt{3}$

Taking greater slope of CE as

$$2 + \sqrt{3}$$

$$y - 1 = (2 + \sqrt{3})(x - 1)$$

Put $y = 0$

$$-1 = (2 + \sqrt{3})(x - 1)$$

$$\frac{-1}{2 + \sqrt{3}} \times \left(\frac{2 - \sqrt{3}}{2 - \sqrt{3}} \right) = x - 1$$

$$x - 1 = \sqrt{3} - 1$$

$$EB = 1 - x = 1 - (\sqrt{3} - 1)$$

$$EB = 2 - \sqrt{3}$$

4. If $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$, then $a + b + c$ is equal to _____.

Official Ans. by NTA (4)

Sol. $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{a \left(1 + x + \frac{x^2}{2!} \dots \right) - b \left(1 - \frac{x^2}{2!} + \dots \right) + c \left(1 - x + \frac{x^2}{2!} \right)}{\left(\frac{x \sin x}{x} \right) x} = 2$$

$$a - b + c = 0 \quad \dots(1)$$

$$a - c = 0 \quad \dots(2)$$

$$\& \frac{a+b+c}{2} = 2$$

$$\Rightarrow \boxed{a+b+c=4}$$

5. The total number of 3×3 matrices A having entries from the set $\{0, 1, 2, 3\}$ such that the sum of all the diagonal entries of AA^T is 9, is equal to _____.

Official Ans. by NTA (766)

Sol. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

diagonal elements of

$$AA^T, \quad a^2 + b^2 + c^2, \quad d^2 + e^2 + f^2, \quad g^2 + h^2 + i^2$$

$$\text{Sum} = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 9$$

$$a, b, c, d, e, f, g, h, i \in \{0, 1, 2, 3\}$$

	Case	No. of Matrices
(1)	All - 1s	$\frac{9!}{9!} = 1$
(2)	One \rightarrow 3 remaining-0	$\frac{9!}{1! \times 8!} = 9$
(3)	One-2 five-1s three-0s	$\frac{9!}{1! \times 5! \times 3!} = 8 \times 63$
(4)	two - 2's one-1 six-0's	$\frac{9!}{2! \times 6!} = 63 \times 4$

Total no. of ways = $1 + 9 + 8 \times 63 + 63 \times 4$
 $= \boxed{766}$

6. Let

$$P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega + 1 \end{bmatrix}$$

where $\omega = \frac{-1 + i\sqrt{3}}{2}$, and I_3 be the identity

matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to _____ .

Official Ans. by NTA (36)

Sol. Let $M = (P^{-1}AP - I)^2$
 $= (P^{-1}AP)^2 - 2P^{-1}AP + I$
 $= P^{-1}A^2P - 2P^{-1}AP + I$
 $PM = A^2P - 2AP + P$
 $= (A^2 - 2A.I + I^2)P$
 $\Rightarrow \text{Det}(PM) = \text{Det}((A - I)^2 \times P)$
 $\Rightarrow \text{Det}P \cdot \text{Det}M = \text{Det}(A - I)^2 \times \text{Det}(P)$
 $\Rightarrow \text{Det} M = (\text{Det}(A - I))^2$

Now $A - I = \begin{bmatrix} 1 & 7 & \omega^2 \\ -1 & -\omega - 1 & 1 \\ 0 & -\omega & -\omega \end{bmatrix}$

$\text{Det}(A - I) = (\omega^2 + \omega + \omega) + 7(-\omega) + \omega^3 = -6\omega$
 $\text{Det}((A - I)^2) = 36\omega^2$
 $\Rightarrow \alpha = 36$

7. If the normal to the curve

$$y(x) = \int_0^x (2t^2 - 15t + 10) dt \text{ at a point } (a, b) \text{ is}$$

parallel to the line $x + 3y = -5$, $a > 1$, then the value of $la + 6bl$ is equal to _____ .

Official Ans. by NTA (406)

Sol. $y(x) = \int_0^x (2t^2 - 15t + 10) dt$

$$y'(x) \Big|_{x=a} = [2x^2 - 15x + 10]_a = 2a^2 - 15a + 10$$

Slope of normal = $-\frac{1}{3}$

$$\Rightarrow 2a^2 - 15a + 10 = 3 \Rightarrow a = 7$$

& $a = \frac{1}{2}$ (rejected)

$$b = y(7) = \int_0^7 (2t^2 - 15t + 10) dt$$

$$= \left[\frac{2t^3}{3} - \frac{15t^2}{2} + 10t \right]_0^7$$

$$\Rightarrow 6b = 4 \times 7^3 - 45 \times 49 + 60 \times 7$$

$$la + 6bl = 406$$

8. Let the curve $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} = 2(x+1)$. If the

numerical value of area bounded by the curve $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to _____ .

Official Ans. by NTA (2)

Sol. $\frac{dy}{dx} = 2(x+1)$

$$\Rightarrow \int dy = \int 2(x+1) dx$$

$$\Rightarrow y(x) = x^2 + 2x + C$$

$$\text{Area} = \frac{4\sqrt{8}}{3}$$

$$-1 + \sqrt{1-C}$$

$$\Rightarrow 2 \int_{-1}^{-1+\sqrt{1-C}} (-(x+1)^2 - C + 1) dx = \frac{4\sqrt{8}}{3}$$

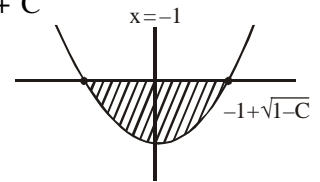
$$\Rightarrow 2 \left[-\frac{(x+1)^3}{3} - Cx + x \right]_{-1}^{-1+\sqrt{1-C}} = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow -(\sqrt{1-C})^3 + 3c - 3C\sqrt{1-C}$$

$$-3 + 3\sqrt{1-C} - 3C + 3 = 2\sqrt{8}$$

$$\Rightarrow C = -1$$

$$\Rightarrow f(x) = x^2 + 2x - 1, f(1) = 2$$



9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x) + f(x + 1) = 2$, for all $x \in \mathbb{R}$. If

$$I_1 = \int_0^8 f(x)dx \text{ and } I_2 = \int_{-1}^3 f(x)dx, \text{ then the value}$$

of $I_1 + 2I_2$ is equal to _____ .

Official Ans. by NTA (16)

Sol. $f(x) + f(x + 1) = 2$
 $\Rightarrow f(x)$ is periodic with period = 2

$$I_1 = \int_0^8 f(x)dx = 4 \int_0^2 f(x)dx$$

$$= 4 \int_0^1 (f(x) + f(1+x))dx = 8$$

Similarly $I_2 = 2 \times 2 = 4$

$$I_1 + 2I_2 = 16$$

10. Let z and w be two complex numbers such that

$$w = z\bar{z} - 2z + 2, \left| \frac{z+i}{z-3i} \right| = 1 \text{ and } \operatorname{Re}(w) \text{ has}$$

minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which w^n is real, is equal to _____.

Official Ans. by NTA (4)

Sol. $w = z\bar{z} - 2z + 2$

$$\left| \frac{z+i}{z-3i} \right| = 1$$

$$\Rightarrow |z+i| = |z-3i|$$

$$\Rightarrow z = x + i, x \in \mathbb{R}$$

$$w = (x+i)(x-i) - 2(x+i) + 2$$

$$= x^2 + 1 - 2x - 2i + 2$$

$$\operatorname{Re}(w) = x^2 - 2x + 3$$

For min $(\operatorname{Re}(w))$, $x = 1$

$$\Rightarrow w = 2 - 2i = 2(1 - i) = 2\sqrt{2} e^{-i\frac{\pi}{4}}$$

$$w^n = (2\sqrt{2})^n e^{-i\frac{n\pi}{4}}$$

For real & minimum value of n ,

$$n = 4$$

